Statistics 330/600b **Due: Thursday 17 February**
2000: Sheet 5

Please attempt at least the starred problems.

*(5.1) (postponed to next week) Let μ and ν be finite measures on $\mathcal{B}(\mathbb{R})$. Define distribution functions $F(t) = \mu(-\infty, t]$ and $G(t) = \nu(-\infty, t]$.

(i) Show that $\mu\{x\} > 0$ for at most countably many *atoms* $x \in \mathbb{R}$.

(ii) Show that $\mu^t G(t) + \nu^t F(t) = \mu(\mathbb{R})\nu(\mathbb{R}) + \sum_i \mu\{x_i\}\nu\{x_i\}$, where $\{x_i : i \in \mathbb{N}\}$ contains all the atoms for both measures.

(iii) Explain how (ii) is related to the integration-by-parts formula:

$$\int F(t)\frac{dG(t)}{dt}\,dt = F(\infty)G(\infty) - \int G(t)\frac{dF(t)}{dt}\,dt$$

*(5.2) In the Notes we needed the function $g(x, y) = py^{p-1}\{f(x) > y\}$ to be product measurable if f is $\mathcal{A}$-measurable. Prove this fact by following these steps.

(i) The map $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\phi(s, t) = \{s > t\}$ is $\mathcal{B}(\mathbb{R}^2)\backslash\mathcal{B}(\mathbb{R})$-measurable. (What do you know about inverse images for ϕ?)

(ii) Prove that the map $(x, y) \mapsto \big(f(x), y\big)$ is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})\backslash\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable.

(iii) Show that the composition of measurable functions is measurable, if the various sigma-fields fit together in the right way.

(iv) Complete the argument.

(5.3) (Kolmogorov's zero-one law) Let $\{\mathcal{G}_i : i \in \mathbb{N}\}$ be a sequence of independent sigma-fields. For each n define $\mathcal{H}_n = \sigma(\mathcal{G}_i : i > n)$. The ***tail sigma-field*** is defined as $\mathcal{H}_\infty = \cap_n \mathcal{H}_n$. Show that, for each H in $\mathcal{H}_\infty$, either $\mathbb{P}H = 0$ or $\mathbb{P}H = 1$, by following these steps.

(i) Show that $\mathcal{H}_\infty$ is a sigma-field.

(ii) Show that $\mathcal{H}_n$ is generated by the class of all finite intersections $G_{n+1}G_{n+2}\ldots G_{n+k}$, with $G_i \in \mathcal{G}_i$, for $k = 1, 2, \ldots$. Deduce via (5.1) that $\mathcal{H}_n$ is independent of the sigma-field $\mathcal{F}_n$ generated by $\cup_{i\le n}\mathcal{G}_i$.

(iii) Deduce that $\mathcal{H}_\infty$ is independent of each $\mathcal{F}_n$, and hence is independent of the sigma-field $\mathcal{F}_\infty$ generated by all the $\mathcal{F}_n$.

(iv) Show that $\mathcal{H}_\infty \subseteq \mathcal{F}_\infty$. Deduce that if $H \in \mathcal{H}_\infty$ then H is independent of itself: $\mathbb{P}(HH) = (\mathbb{P}H)(\mathbb{P}H)$.

(5.4) Let $\bar{Z}_n = (Z_1 + \ldots + Z_n)/n$, for a sequence $\{Z_i\}$ of independent random variables.

(i) Use the Kolmogorov zero-one law from the previous Problem to show that the set $\{\limsup \bar{Z}_n > r\}$ is a tail event for each constant r, and hence it has probability either zero or one. Deduce that $\limsup \bar{Z}_n = c_0$ almost surely, for some constant c_0 (possibly $\pm\infty$).

(ii) If $\bar{Z}_n$ converges to a finite limit (possibly random) at each ω in a set A with $\mathbb{P}A > 0$, show that in fact there must exists a finite constant c_0 for which $\bar{Z}_n \to c_0$ almost surely.

(5.5) Let P and Q be finite measures defined on the same sigma-field $\mathcal{F}$, with densities p and q with respect to a measure μ. Suppose $\mathcal{X}_0$ is a measurable set with the property that there exists a nonnegative constant K such that $q \ge Kp$ on $\mathcal{X}_0$ and $q \le Kp$ on $\mathcal{X}_0^c$. For each $\mathcal{F}$-measurable function with $0 \le f \le 1$ and $Pf \le P\mathcal{X}_0$, prove that $Qf \le Q\mathcal{X}_0$. Hint: Prove that $(q - Kp)(2\mathcal{X}_0 - 1) \ge (q - Kp)(2f - 1)$, then integrate. *To statisticians this result is known as the Neyman-Pearson Lemma.*