Statistics 330/600b **Due: Thursday 24 February**
2000: Sheet 6

Please attempt at least the starred problems.

*(6.1) Let μ and ν be finite measures on $\mathcal{B}(\mathbb{R})$. Define distribution functions $F(t) = \mu(-\infty, t]$ and $G(t) = \nu(-\infty, t]$.

(i) Show that $\mu\{x\} > 0$ for at most countably many *atoms* $x \in \mathbb{R}$.

(ii) Show that $\mu^t G(t) + \nu^t F(t) = \mu(\mathbb{R})\nu(\mathbb{R}) + \sum_i \mu\{x_i\}\nu\{x_i\}$, where $\{x_i : i \in \mathbb{N}\}$ contains all the atoms for both measures.

(iii) Explain how (ii) is related to the integration-by-parts formula:

$$\int F(t)\frac{dG(t)}{dt}\,dt = F(\infty)G(\infty) - \int G(t)\frac{dF(t)}{dt}\,dt$$

*(6.2) Let $B_1, B_2, \ldots$ be independent events for which $\sum_{i=1}^{\infty} \mathbb{P}B_i = \infty$ Show that $\mathbb{P}\{B_i \text{ infinitely often }\} = 1$ by following these steps.

(i) Show that $\mathbb{P}B_1^c B_2^c \ldots B_n^c \le \exp\left(-\sum_{i=1}^{n} \mathbb{P}B_i\right) \to 0$.

(ii) Deduce that $\prod_{i=1}^{\infty} B_i^c = 0$ almost surely.

(iii) Deduce that $\sum_{i=1}^{\infty} B_i \ge 1$ almost surely.

(iv) Deduce that $\sum_{i=m}^{\infty} B_i \ge 1$ almost surely, for each finite m. Hint: The events $B_m, B_{m+1}, \ldots$ are independent.

(v) Complete the proof.

Remark: This result is a converse to the Borel-Cantelli Lemma discussed in Section 2.6*. A stronger converse was established in the Problems to Chapter 2.*

*(6.3) Let $X_1, X_2, \ldots$ be independent, identically distributed, random variables with $\mathbb{P}|X_i| = \infty$. Let $S_n = X_1 + \ldots + X_n$. Show that S_n/n cannot converge almost surely to a finite value. Hint: If $S_n/n \to c$, show that $(S_{n+1} - S_n)/n \to 0$ almost surely. Deduce from Problem (6.2) that $\sum_n \mathbb{P}\{|X_n| \ge n\} < \infty$. Argue for a contradiction by showing that $\mathbb{P}|X_1| \le 1 + \sum_{n=1}^{\infty} \mathbb{P}\{|X_n| \ge n\}$.

*(6.4) Let λ and μ both denote counting measure on the sigma-field of all subsets of $\mathbb{N}$. Let $f(x, y) = \{y = x\} - \{y = x + 1\}$. Show that $\mu^x \lambda^y f(x, y) = 0$ but $\lambda^y \mu^x f(x, y) = 1$. Why does the Fubini Theorem not apply?

(6.5) For nonnegative random variables $Z_1, \cdots, Z_m$, show that

$$\mathbb{P}Z_1 Z_2 \cdots Z_m \le \int_0^1 Q_{Z_1}(u) Q_{Z_2}(u) \cdots Q_{Z_m}(u)\,du,$$

with Q_{Z_i} the quantile function for Z_i.

(6.6) Let X and Y be independent, real-valued ranom variables for which $\mathbb{P}(XY)$ is well defined, that is, either $\mathbb{P}(XY)^+ < \infty$ or $\mathbb{P}(XY)^- < \infty$. Suppose neither X nor Y is degenerate (equal to zero almost surely). Show that both $\mathbb{P}X$ and $\mathbb{P}Y$ are well defined and $\mathbb{P}(XY) = (\mathbb{P}X)(\mathbb{P}Y)$. Hint: What would you learn from $\infty > \mathbb{P}(XY)^+ = \mathbb{P}X^+Y^+ + \mathbb{P}X^-\mathbb{P}Y^-$?