Contents

PREFACE xi

CHAPTER 1: MOTIVATION
§1 Why bother with measure theory? 1
§2 The cost and benefit of rigor 3
§3 Where to start: probabilities or expectations? 5
§4 The de Finetti notation 7
*§5 Fair prices 11
§6 Problems 13
§7 Notes 14

CHAPTER 2: A MODICUM OF MEASURE THEORY
§1 Measures and sigma-fields 17
§2 Measurable functions 22
§3 Integrals 26
*§4 Construction of integrals from measures 29
§5 Limit theorems 31
§6 Negligible sets 33
*§7 L^p spaces 36
*§8 Uniform integrability 37
§9 Image measures and distributions 39
§10 Generating classes of sets 41
*§11 Generating classes of functions 43
§12 Problems 45
§13 Notes 51

CHAPTER 3: DENSITIES AND DERIVATIVES
§1 Densities and absolute continuity 53
*§2 The Lebesgue decomposition 58
§3 Distances and affinities between measures 59
§4 The classical concept of absolute continuity 65
*§5 Vitali covering lemma 68
*§6 Densities as almost sure derivatives 70
§7 Problems 71
§8 Notes 75

CHAPTER 4: PRODUCT SPACES AND INDEPENDENCE
§1 Independence 77
§2 Independence of sigma-fields 80
§3 Construction of measures on a product space 83
§4 Product measures 88
*§5 Beyond sigma-finiteness 93
§6 SLLN via blocking 95
*§7 SLLN for identically distributed summands 97
*§8 Infinite product spaces 99

§9 Problems 102
§10 Notes 108

CHAPTER 5: CONDITIONING
§1 Conditional distributions: the elementary case 111
§2 Conditional distributions: the general case 113
§3 Integration and disintegration 116
§4 Conditional densities 118
*§5 Invariance 121
§6 Kolgomorov's abstract conditional expectation 123
*§7 Sufficiency 128
§8 Problems 131
§9 Notes 135

CHAPTER 6: MARTINGALE ET AL
§1 What are they? 138
§2 Stopping times 142
§3 Convergence of positive supermartingales 147
§4 Convergence of submartingales 151
*§5 Proof of the Krickeberg decomposition 152
*§6 Uniform integrability 153
*§7 Reversed martingales 155
*§8 Symmetry and exchangeability 159
§9 Problems 162
§10 Notes 166

CHAPTER 7: CONVERGENCE IN DISTRIBUTION
§1 Definition and consequences 169
§2 Lindeberg's method for the central limit theorem 176
§3 Multivariate limit theorems 181
§4 Stochastic order symbols 182
*§5 Weakly convergent subsequences 184
§6 Problems 186
§7 Notes 190

CHAPTER 8: FOURIER TRANSFORMS
§1 Definitions and basic properties 193
§2 Inversion formula 195
§3 A mystery? 198
§4 Convergence in distribution 198
*§5 A martingale central limit theorem 200
§6 Multivariate Fourier transforms 202
*§7 Cramér-Wold without Fourier transforms 203
*§8 The Lévy-Cramér theorem 205
§9 Problems 206
§10 Notes 208

CHAPTER 9: BROWNIAN MOTION
§1 Prerequisites 211
§2 Brownian motion and Wiener measure 213
§3 Existence of Brownian motion 215
*§4 Finer properties of sample paths 217
§5 Strong Markov property 219
*§6 Martingale characterizations of Brownian motion 222
*§7 Functionals of Brownian motion 226
*§8 Option pricing 228
§9 Problems 230
§10 Notes 234

CHAPTER 10: REPRESENTATIONS AND COUPLINGS
§1 What is coupling? 237
§2 Almost sure representations 239
*§3 Strassen's Theorem 242
*§4 The Yurinskii coupling 244
§5 Quantile coupling of Binomial with normal 248
§6 Haar coupling—the Hungarian construction 249
§7 The Komlós-Major-Tusnády coupling 252
§8 Problems 256
§9 Notes 258

CHAPTER 11: EXPONENTIAL TAILS AND THE LAW OF THE ITERATED LOGARITHM
§1 LIL for normal summands 261
§2 LIL for bounded summands 264
*§3 Kolmogorov's exponential lower bound 266
*§4 Identically distributed summands 268
§5 Problems 271
§6 Notes 272

CHAPTER 12: MULTIVARIATE NORMAL DISTRIBUTIONS
§1 Introduction 274
*§2 Fernique's inequality 275
*§3 Proof of Fernique's inequality 276
§4 Gaussian isoperimetric inequality 278
*§5 Proof of the isoperimetric inequality 280
§6 Problems 285
§7 Notes 287

APPENDIX A: MEASURES AND INTEGRALS
§1 Measures and inner measure 289
§2 Tightness 291
§3 Countable additivity 292
§4 Extension to the $\cap c$-closure 294
§5 Lebesgue measure 295
§6 Integral representations 296
§7 Problems 300
§8 Notes 300

APPENDIX B: HILBERT SPACES
§1 Definitions 301
§2 Orthogonal projections 302
§3 Orthonormal bases 303
§4 Series expansions of random processes 305
§5 Problems 306
§6 Notes 306

APPENDIX C: CONVEXITY
§1 Convex sets and functions 307
§2 One-sided derivatives 308
§3 Integral representations 310
§4 Relative interior of a convex set 312
§5 Separation of convex sets by linear functionals 313
§6 Problems 315
§7 Notes 316

APPENDIX D: BINOMIAL AND NORMAL DISTRIBUTIONS
§1 Tails of the normal distributions 317
§2 Quantile coupling of Binomial with normal 320
§3 Proof of the approximation theorem 324
§4 Notes 328

APPENDIX E: MARTINGALES IN CONTINUOUS TIME
§1 Filtrations, sample paths, and stopping times 329
§2 Preservation of martingale properties at stopping times 332
§3 Supermartingales from their rational skeletons 334
§4 The Brownian filtration 336
§5 Problems 338
§6 Notes 338

APPENDIX F: DISINTEGRATION OF MEASURES
§1 Representation of measures on product spaces 339
§2 Disintegrations with respect to a measurable map 342
§3 Problems 343
§4 Notes 345

INDEX 347