Statistics 330/600 **Due: Thursday 19 January**
2006: Sheet 1

Please attempt at least the starred problems.

*(1.1) Please send to me (david.pollard@yale.edu) an email giving your year, your major or graduate department, and noting whether you have taken (or are in the process of taking) real analysis, measure theory, or any probability course.

*(1.2) Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space. Suppose $\{A_i : i \in \mathbb{N}\} \subset \mathcal{A}$ with $A_i \uparrow A$ as $i \to \infty$. (That is, $A_i \subseteq A_{i+1}$ for all i and $A = \cup_{i \in \mathbb{N}} A_i$.)

(i) Show that $A = A_1 \cup \bigcup_{i \in \mathbb{N}} (A_{i+1} \backslash A_i)$. [Remember: $E \backslash F = E \cap F^c$.]

(ii) Deduce that $\mu A_i \uparrow \mu A$.

(iii) Suppose $\{B_i : i \in \mathbb{N}\} \subset \mathcal{A}$ with $B_i \downarrow B$ as $i \to \infty$. Show that $B_1 \backslash B_i \uparrow B_1 \backslash B$. If $\mu B_1 < \infty$, deduce that $\mu B_i \downarrow \mu B$.

*(1.3) Let μ be a finite measure on the Borel sigma-field $\mathcal{B}(\mathcal{X})$ of a metric space $\mathcal{X}$. [You may take $\mathcal{X} = \mathbb{R}^2$ if you are not yet comfortable with general metric spaces.] Show that for each $\epsilon > 0$ and each $B \in \mathcal{B}(\mathcal{X})$ there exists a closed set F_ϵ and an open set G_ϵ for which $F_\epsilon \subseteq B \subseteq G_\epsilon$ and $\mu(G_\epsilon \backslash F_\epsilon) < \epsilon$.

(1.4) Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For $B \subseteq \mathcal{Y}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. For $A \subseteq \mathcal{X}$ define $T(A) := \{T(x) : x \in A\}$. Some of the following assertions are true and some are false.

$$T\left(\cup_i A_i\right) = \cup_i T(A_i) \quad \text{and} \quad T^{-1}\left(\cup_i B_i\right) = \cup_i T^{-1}(B_i)$$
$$T\left(\cap_i A_i\right) = \cap_i T(A_i) \quad \text{and} \quad T^{-1}\left(\cap_i B_i\right) = \cap_i T^{-1}(B_i)$$
$$T\left(A^c\right) = \left(T\left(A\right)\right)^c \quad \text{and} \quad T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c$$
$$T^{-1}\left(T(A)\right) = A \quad \text{and} \quad T\left(T^{-1}(B)\right) = B$$

Provide counterexamples for each of the false assertions.

(1.5) The set $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is called the ***extended real line***. Write $\mathcal{A}$ for the sigma-field on $\overline{\mathbb{R}}$ generated by $\mathcal{B}(\mathbb{R})$ together with the two singleton sets $\{-\infty\}$ and $\{\infty\}$.

(i) Show that $\mathcal{A}$ is also generated by $\mathcal{E} = \{\,[-\infty, t] : t \in \mathbb{R}\}$.

(ii) Let $\psi : [-1, +1] \to \overline{\mathbb{R}}$ be defined by $\psi(-1) = -\infty$ and $\psi(+1) = +\infty$ and

$$\psi(x) = \frac{x}{1 - |x|} \qquad \text{for } |x| < 1.$$

Show that $B \in \mathcal{A}$ if and only if $\psi^{-1}(B) \in \mathcal{B}[0, 1]$.

(1.6) Follow these steps to show that there cannot exist a translation invariant (countably additive) probability measure defined for the collection of all subsets of $(0, 1]$. For a and y in $(0, 1]$ define

$$x \oplus y = \begin{cases} x + y & \text{if } x + y \le 1 \\ x + y - 1 & \text{if } x + y > 1 \end{cases}.$$

For $A \subseteq (0, 1]$ and $x \in (0, 1]$ define $A \oplus x = \{y \oplus x : y \in A\}$. Suppose μ is a measure that is defined for all subsets of $(0, 1]$ for which $\mu\left(A \oplus x\right) = \mu A$ for all A and x. Define $\mathcal{Q} = \{q \in (0, 1] : q \text{ is rational }\}$. Define an equivalence relation on $(0, 1]$ by $x \sim y$ if $x = y \oplus r$ for some $r \in \mathcal{R}$. Let A be a set that contains exactly one point from each equivalence class.

(i) Show that $A \oplus r$ and $A \oplus s$ are disjoint sets if r and s are distinct elements of $\mathcal{Q}$ and that $(0, 1] = \cup_{r \in \mathcal{Q}} \left(A \oplus r\right)$. Hint: The argument is much neater if you identify $(0, 1]$ with the perimeter of a circle of circumference 1.

(ii) Deduce that $\mu(0, 1] = \sum_{r \in \mathcal{Q}} \mu\left(A \oplus r\right)$.

(iii) Show that

$$\mu(0, 1] = \begin{cases} \infty & \text{if } \mu A > 0 \\ 0 & \text{if } \mu A = 0 \end{cases}.$$

Deduce that μ cannot be a probability measure (or any other non-trivial, finite measure).