Statistics 330/600 **Due: Thursday 6 April**
2006: Sheet 10

*(10.1) Suppose μ and $\mathbb{P}$ are probability measures on a ***countably generated*** sigma-field $\mathcal{F}_\infty$ on a set Ω. That is, $\mathcal{F}_\infty = \sigma\{F_i : i \in \mathbb{N}\}$. Let π_n denote the partition of Ω generated by the sets $F_1, \dots, F_n$. (That is, each nonempty member of π_n is of the form $\cap_{i \le n} A_i$ with $A_i = F_i$ or $A_i = F_i^c$.) Define

$$X_n(\omega) = \sum\nolimits_{B \in \pi_n} \{\omega \in B, \mathbb{P}B > 0\} \frac{\mu B}{\mathbb{P}B}$$
$$Z_n(\omega) = \sum\nolimits_{B \in \pi_n} \{\omega \in B, \lambda B > 0\} \frac{\mu B}{\lambda B} \qquad \text{where } \lambda = \tfrac{1}{2}(\mu + \mathbb{P}).$$

(i) Define $\Omega_n = \cup\{B \in \pi_n : \mathbb{P}B > 0\}$. Show that $\mathbb{P}\Omega_n = 1$ for each n.

(ii) Define $\mathcal{F}_n = \sigma(\pi_n) = \sigma\{F_i : i = 1, \dots, n\}$. Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a $\mathbb{P}$-supermartingale, which converges $\mathbb{P}$ almost surely to some random variable X_∞.

(iii) Show that $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a λ-martingale, which converges λ almost surely to a random variable Z_∞. Show also that Z_∞ is a version of the density $d\mu/d\lambda$.

(iv) Show that there is an $\mathcal{F}_\infty$-measurable subset Ω_0 of $\cap_{n \in \mathbb{N}} \Omega_n$ with $\mathbb{P}\Omega_0 = 1$ such that

$$X_n(\omega) \to X_\infty(\omega) < \infty \qquad \text{and} \qquad \frac{2X_n(\omega)}{1 + X_n(\omega)} = Z_n(\omega) \qquad \text{for each } \omega \in \Omega_0.$$

(v) Deduce that

$$\frac{2X_\infty}{1 + X_\infty}\Omega_0 = Z_\infty \Omega_0 \qquad \text{a.e. } [\lambda]$$

(vi) Deduce that

$$\mathbb{P}\frac{X_\infty \Omega_0 F}{1 + X_\infty} = \mu \frac{\Omega_0 F}{1 + X_\infty} \qquad \text{for each } F \in \mathcal{F}_\infty.$$

(vii) Deduce that

$$\mathbb{P}X_\infty \Omega_0 f = \mu \Omega_0 f \qquad \text{for each } f \in \mathcal{M}^+(\mathcal{F}_\infty).$$

(viii) Conclude that X_∞ is a version of the density $d\mu_0/d\mathbb{P}$, where μ_0 is the part of μ that is absolutely continuous with respect to $\mathbb{P}$.

*(10.2) For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ for which either $\mathbb{P}G = 0$ or $\mathbb{P}G = 1$ for every $G \in \mathcal{G}$.

(i) For each $Z \in \mathcal{M}^+(\mathcal{G})$, show that there exists a constant $c \in [0, \infty]$ for which $\mathbb{P}\{Z = c\} = 1$. Hint: Consider $\sup\{t \in \mathbb{R} : \mathbb{P}\{Z \ge t\} = 1$.

(ii) For each $X \in \mathcal{M}^+(\mathcal{F})$, show that $\mathbb{P}(X \mid \mathcal{G}) = \mathbb{P}X$ a.e. $[\mathbb{P}]$.

(10.3) (generalized STL for positive supermartingales) UGMTP Problem 6.6. Hint: Start from what you know about $\sigma \wedge n$ and $\tau \wedge n$ then justify a passage to the limit.

(10.4) UGMTP Problem 6.4. You might prefer to prove the assertion only for $X \in \mathcal{M}^+$.