Statistics 330/600 **Due: Thursday 13 April**
2006: Sheet 11

*(11.1) (closure and interior) UGMTP Problem 7.3.

*(11.2) (CMT via Fubini) UGMTP Problem 7.10.

*(11.3) Suppose the random k-vectors $\{X_n : n \in \mathbb{N}\}$ converge in distribution to X. Let $\{A_n\}$ be a sequence of random $k \times k$ matrices that converges in probability to a nonrandom matrix A, and let $\{B_n\}$ be a sequence of random k-vectors that converges in probability to a nonrandom vector B. Show that $A_n X_n + B_n \rightsquigarrow AX + B$. Hint: Consider functions of the random object (X_n, A_n, B_n), which you might like to rearrange as a random $(k + k^2 + k)$-vector.

*(11.4) (cid of MVN) UGMTP Problem 7.12, part (ii) only.

(11.5) Suppose $\psi_i : \mathcal{Y} \to \mathcal{Z}_i$ and $\mathcal{Z}_i$ is equipped with a sigma-field $\mathcal{C}_i$, for each i in some index set $\mathcal{I}$. Define $\mathcal{B}$ as the smallest sigma-field on $\mathcal{Y}$ for which each ψ_i is $\mathcal{B}\backslash\mathcal{C}_i$-measurable.

(i) Show that $\mathcal{B}$ is generated by the collection of sets $\cup_{i\in\mathcal{I}}\mathcal{E}_i$, where $\mathcal{E}_i := \{\psi_i^{-1}(C) : C \in \mathcal{C}_i\}$.

(ii) Suppose $T : \Omega \to \mathcal{Y}$ and that $\mathcal{F}$ is a sigma-field on Ω. Show that T is $\mathcal{F}\backslash\mathcal{B}$-measurable if and only if $\psi_i \circ T$ is $\mathcal{F}\backslash\mathcal{C}_i$-measurable for each $i \in \mathcal{I}$.

(iii) Specialize to the case where $\mathcal{I} = \mathbb{N}$ and $\mathcal{Y} = \mathbb{R}^{\mathbb{N}}$, with ψ_i as the ith coordinate map: if $x = (x_i : i \in \mathbb{N}) \in \mathbb{R}^{\mathbb{N}}$ then $\psi_i(x) = x_i$. Suppose $T(\omega) = (X_1(\omega), X_2(\omega), \ldots)$ for some sequence of $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable real random variables $X_1, X_2, \ldots$. Show that T is $\mathcal{F}\backslash\mathcal{B}$-measurable.

(iv) Show that $\sigma(T) = \{T^{-1}(B) : B \in \mathcal{B}\}$.

(v) Show that $\sigma(T) = \mathcal{F}_\infty$, the smallest sigma-field on Ω for which X_i is $\mathcal{F}_\infty\backslash\mathcal{B}(\mathbb{R})$-measurable for each $i \in \mathbb{N}$.

(vi) Show that the elements of $\mathcal{M}^+(\Omega, \mathcal{F}_\infty)$ are precisely those functions of the form $g(T(\omega))$ for some $g \in \mathcal{M}^+(\mathbb{R}^{\mathbb{N}}, \mathcal{B})$. Hint: Start with the bounded random variables.