Statistics 330/600 **Due: Thursday 20 April**
2006: Sheet 12

Please do not work as teams for this final homework sheet. Partial or incomplete solutions are acceptable.

*(12.1) Suppose $\{X_n : n \in \mathbb{N}_0\}$ and $\{Y_n : n \in \mathbb{N}_0\}$ are both martingales for the same filtration $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$. Suppose $\sigma : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is a random variable for which $X_n = Y_n$ on the set $\{\sigma = n\}$, for each n in $\mathbb{N}_0$. Define $Z_n = X_n\{\sigma \le n\} + Y_n\{\sigma > n\}$.

(i) If σ is a stopping time, show that $\{Z_n : n \in \mathbb{N}_0\}$ is a martingale.

(ii) Suppose $Y_n \equiv 0$ and $\{X_n\}$ is a positive martingale, which converges almost surely to a random variable X_∞ with $\mathbb{P}\{X_\infty > 0\} > 0$. Define $\sigma = \sup\{n : X_n = 0\}$. Show that $\{Z_n\}$ is not a martingale.

*(12.2) (CLT without finite moments) UGMTP Problem 7.21.

*(12.3) Suppose μ is a finite measure on $\mathcal{B}(\mathbb{R})$ and $\kappa(\cdot)$ is an infinitely differentiable function which is zero outside the interval $(-1, +1)$. Show that the function $g(t) = \mu^x \kappa(x + t)$ is also infinitely differentiable.

*(12.4) Suppose $\{X_n : n \in \mathbb{N}\}$ is a sequence of integer-valued random variables for which there exist constants $a_n \in \mathbb{R}$ and $b_n > 0$ such that $Z_n := (X_n - a_n)/b_n \rightsquigarrow N(0, 1)$. Show that we must have $b_n \to \infty$. Hint: For each $\delta > 0$ show that both $\mathbb{P}\{1 < Z_n \le 1 + \delta\}$ and $\mathbb{P}\{1 + \delta < Z_n \le 1 + 2\delta\}$ must converge to nonzero values. What does that tell you about b_n?

*(12.5) Suppose P and P_n, for $n \in \mathbb{N}$, are probability measures on $\mathcal{B}(\mathbb{R}^2)$. Let $\mathcal{R}$ denote the set of all rectangles $(a, b] \times (c, d]$. Suppose $P_n R \to P R$ for each R in the set $\mathcal{R}_P = \{R \in \mathcal{R} : P(\partial R) = 0\}$. Show that $P_n \rightsquigarrow P$ by expanding on the following sketch of a proof.

(i) Show that both the sets $\{x \in \mathbb{R} : P\left(\{x\} \times \mathbb{R}\right) > 0\}$ and $\{y \in \mathbb{R} : P\left(\mathbb{R} \times \{y\}\right) > 0\}$ are, at worst, countably infinite.

(ii) Show that there is a countable subset $\mathcal{R}_0$ of $\mathcal{R}_P$ with the property that

$$G = \cup\{R \in \mathcal{R}_0 : G \supset R \in \mathcal{R}_0\}$$

for every open subset G of $\mathbb{R}^2$.

(iii) Show that $P_n D \to P D$ for every set D expressible as a finite union of $\mathcal{R}_0$ sets.

(iv) Deduce that $\liminf_n P_n G \ge P G$ for each open G.

(v) Deduce that $P_n D \to P D$ for each Borel set D with $P\left(\partial D\right) = 0$.

(vi) What then?