Statistics 330/600 **Due: Thursday 2 February**
2006: Sheet 3

Please attempt at least the starred problems.

*(3.1) (separation of pairs of points) UGMTP Problem 2.5. Then explain why $\mathcal{E} = \{(-\infty, t] : t \in \mathbb{R}\}$ does not generate $\mathcal{B}[-\infty, \infty]$.

*(3.2) Suppose $f, g, h \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$ with g and h nonnegative. Suppose we have only defined the integral on $\mathcal{M}^+(\mathcal{X}, \mathcal{A}, \mu)$ and wish to extend it to $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$.

(i) Suppose $f = g - h$. Show that $\mu f^+ - \mu f^- = \mu g - \mu h$. Hint: Consider $h + f^+$.

(ii) Show that $\mu f := \mu f^+ - \mu f^-$ defines μ as an increasing linear functional on $\mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$.

(iii) If $f_1, f_2 \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$, show that $|\mu f_1 - \mu f_2| \le \mu |f_1 - f_2|$. Hint: Consider $f_2 + |f_1 - f_2|$.

*(3.3) (cgce in prob) UGMTP Problem 2.14. You should assume that X and each X_n take only real values. For part (i), please do not just reproduce the textbook proof. For part (ii), you might consider indicator functions of sets A_n with $\mathbb{P}A_n = 1/n$, for some appropriate probability space.

(3.4) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of real-valued random variables, each defined on Ω and each with the same distribution P. That is, P is a probability measure on $\mathcal{B}(\mathbb{R})$ for which $\mathbb{P} f(X_i) = Pf$ at least for each f in $\mathcal{M}^+(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Suppose $P|x| < \infty$. Use Dominated Convergence to show that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{P}|X_i|\{|X_i| > i\} \to 0 \qquad \text{as } n \to \infty.$$

Hint: Express the left-hand side as a P integral.