Statistics 330/600 **Due: Thursday 16 February**
2006: Sheet 5

Please attempt at least the starred problems.

*(5.1) Suppose $\mathcal{E}_i$ is a collection of subsets of $\mathcal{X}_i$ with $\mathcal{X}_i \in \mathcal{E}_i$, for $i = 1, 2$. Show that $\sigma\left(\mathcal{E}_1 \times \mathcal{E}_2\right) = \sigma(\mathcal{E}_1) \otimes \sigma(\mathcal{E}_2)$, as sigma-fields on $\mathcal{X}_1 \times \mathcal{X}_2$.

*(5.2) Let $\mathcal{X}$ be a set equipped with a countably generated sigma-field $\mathcal{A}$. That is, $\mathcal{A} = \sigma(\mathcal{E})$ where $\mathcal{E} = \{E_i : i \in \mathbb{N}\}$. Suppose also that $\{x\} \in \mathcal{A}$ for each x in $\mathcal{X}$. Without loss of generality, suppose $\mathcal{E}$ is stable under complements.

(i) Show that $\{x\} = \cap\{E_i : x \in E_i\}$ for each x in $\mathcal{X}$.

(ii) Show that $\Delta^c := \{(x_1, x_2) \in \mathcal{X}^2 : x_1 \neq x_2\} = \cup_{i \in \mathbb{N}} E_i \times E_i^c \in \mathcal{A} \otimes \mathcal{A}$.

*(5.3) ($2k$th moment) UGMTP Problem 4.24, but only for $k = 3$.

*(5.4) (generalized maximal) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A_1, \ldots, A_n, B_1, \ldots, B_n$ be sets in $\mathcal{F}$ with the following property: for some constant β,

$$\mathbb{P}B_i D \geq \beta \mathbb{P}D \qquad \text{for all } D \in \sigma\{A_1, \ldots, A_i\}, \text{ for each } i.$$

Show that $\beta \mathbb{P} \cup_{i \leq n} A_i \leq \mathbb{P} \cup_{i \leq n} \left(A_i B_i\right)$.

(5.5) (completeness of $\mathcal{L}^p$) UGMTP Problem 2.19.