Statistics 330/600 **Due: Thursday 23 February**
2006: Sheet 6

Please attempt at least the starred problems.

*(6.1) (ϵ–δ on field for absolute continuity) UGMTP Problem 3.6. Don't assume $\mathcal{E}$ is countable. Instead, appeal to the result from Example 2.5.

*(6.2) (Hellinger product) UGMTP Problem 4.18.

*(6.3) (Radon-Nikodym) Suppose μ and ν are finite measures with $\nu \ll \mu$. Modify the argument from UGMTP 2.7 (the Lebesgue Decomposition) to show that ν has a density Δ with respect to μ with $0 \le \Delta(x) < \infty$ for every x. *I want you to give a complete proof. Do not just deduce the result as a special case from UGMTP 2.7.*

(6.4) (Compare with UGMTP Problem 2.19.) For a measure space $(\mathcal{X}, \mathcal{A}, \mu)$, suppose $\{f_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^p(\mu)$, for some fixed $p > 1$, that is, $\|f_n - f_m\|_p \to 0$ as $\min(m, n) \to \infty$. Show that there exists a real-valued function f in $\mathcal{L}^p(\mu)$ for which $\|f_n - f\|_p \to 0$ as $n \to \infty$, by the following steps.

(i) Show that there is no loss of generality in supposing $f_n \ge 0$ for all n. Hint: Consider $f_n^{\pm}$.

(ii) For all nonnegative constants α and β, show that

$$|\alpha - \beta|^p \le |\alpha^p - \beta^p| \le p|\alpha - \beta| \left(\alpha^{p-1} + \beta^{p-1}\right).$$

Hint: Reduce to the case $\alpha > \beta = 1$.

(iii) Show that $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$.

(iv) Show that $\{f_n^p : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^1(\mu)$. Hint: Hölder.

(v) Show that there exists a function g in $\mathcal{L}^p(\mu)$ with $g \ge 0$ and $\mu|f_n^p - g^p| \to 0$.

(vi) Show that $\|f_n - g\|_p \to 0$.

(vii) Anything more to do?