Statistics 330/600 **Due: Thursday 30 March**
2006: Sheet 9

*(9.1) Suppose $\{\xi_i : i \in \mathbb{N}\}$ are independent random variables with $\mathbb{P}\{\xi_i = 1\} = 1/2 = \mathbb{P}\{\xi_i = -1\}$, for each i. Suppose $X_0 \equiv 1$ and $X_n = X_0 + \xi_1 + \ldots + \xi_n$. Let $\mathcal{F}_n = \sigma\{X_0, \xi_1, \ldots, \xi_n\}$. Define

$$\sigma = \inf\{n : X_n = 0\} \qquad \text{and} \qquad \tau = \inf\{n : X_n = 2\} \qquad \text{and} \qquad Z_n = X_{n \wedge \sigma}.$$

(i) Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale. Does $\{X_n\}$ converge almost surely?

(ii) Show that $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a positive martingale, which converges almost surely to some real-valued random variable Z_∞.

(iii) From the facts that $|X_{n+1} - X_n| = 1$ and $Z_{n+1} - Z_n \to 0$ almost surely, deduce that $\sigma < \infty$ almost surely.

(iv) Show that $\mathbb{P}X_\tau\{\tau < \infty\} \neq \mathbb{P}X_0$. Why does this fact not contradict the Stoppingg Time Lemma?

*(9.2) Suppose W and X are nonnegative random variables for which there exists nonnegative constants β and C for which

$$t\mathbb{P}\{W > \beta t\} \leq C\mathbb{P}Z\{W > t\} \qquad \text{for all } t > 0.$$

Show that

$$\|W\|_p \leq \frac{Cp\beta^p}{p-1}\|Z\|_p$$

for each $p > 1$ for which $\mathbb{P}W^p < \infty$.

(9.3) (Doob for $p > 1$) UGMTP Problem 6.9. You do not have to reprove any results established in other Problems on this Sheet.