Comment on second part of Problem 12.1

Suppose $\{X_n : n \in \mathbb{N}_0\}$ and $\{Y_n : n \in \mathbb{N}_0\}$ are both martingales for the same filtration $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$. Suppose $\sigma : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is a random variable for which $X_n = Y_n$ on the set $\{\sigma = n\}$, for each n in $\mathbb{N}_0$. Define $Z_n = X_n\{\sigma \le n\} + Y_n\{\sigma > n\}$.

(i) If σ is a stopping time, show that $\{Z_n : n \in \mathbb{N}_0\}$ is a martingale.

(ii) Suppose $Y_n \equiv 0$ and $\{X_n\}$ is a positive martingale, which converges almost surely to a random variable X_∞ with $\mathbb{P}\{X_\infty > 0\} > 0$. Define $\sigma = \sup\{n : X_n = 0\}$. Show that $\{Z_n\}$ is not a martingale.

Of course, σ is not a stopping time. That in itself is enough to cast doubt on $\{Z_n\}$ being a martingale: if $\{\sigma \le n\} \notin \mathcal{F}_n$ then it is not likely that Z_n is $\mathcal{F}_n$-measurable. However, in this case, there are some trivial reasons for things almost working.

By the Remark on page 48 of UGMTP,

$$\sigma = \infty \qquad \text{almost surely on the set } \Omega_0 := \cup_{i \in \mathbb{N}_0}\{X_i = 0\}.$$

How is σ defined on Ω_o^c? That is, how should we define $\sup \emptyset$? I had required σ to take values in $\mathbb{N}_0 \cup \{\infty\}$. If I took $\sigma = 0$ on Ω_0^c then I would be forcing $X_0 = 0$ and then $Z_n = 0$ almost surely, which, apart from quibbles about negligible sets, makes $\{Z_n\}$ a martingale. Of course, my requirement that X_∞ be nontrivial is then violated.

The whole counterexample is rather silly. I had intended to make the point that $\mathbb{P}Z_n$ need not be a constant if σ is not a stopping time. A better illustration would be a simple random walk: for iid random variables $\xi_1, \xi_2, \dots$ with $\mathbb{P}\{\xi_i = +1\} = \mathbb{P}\{\xi_i = -1\} = 1/2$, let

$$-Y_n = X_n = \xi_1 + \ldots + \xi_n$$

with $X_0 = 0$. Let $\sigma = 2\{X_1 = +1, X_2 = 0\}$. That is, σ takes only the values 0 and 2. Note that $\mathbb{P}Z_0 = 0$ but

$$\mathbb{P}Z_1 = \mathbb{P}\big(\{\sigma = 0\}X_1 - \{\sigma = 2\}X_1\big) = -1/2.$$

Sorry about that.