Statistics 330/600 **Due: Thursday 25 January**
2007: Sheet 1

Please attempt at least the starred problems.

*(1.1) Please send to me (david.pollard@yale.edu) an email stating whether you have taken (or are in the process of taking) real analysis, measure theory, or any probability course.

*(1.2) Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For $B \subseteq \mathcal{Y}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. For $A \subseteq \mathcal{X}$ define $T(A) := \{T(x) : x \in A\}$. Some of the following assertions are true and some are false.

$$
\begin{array}{rcl}
T\left(\cup_i A_i\right) = \cup_i T(A_i) & \text{and} & T^{-1}\left(\cup_i B_i\right) = \cup_i T^{-1}(B_i) \\
T\left(\cap_i A_i\right) = \cap_i T(A_i) & \text{and} & T^{-1}\left(\cap_i B_i\right) = \cap_i T^{-1}(B_i) \\
T\left(A^c\right) = \left(T\left(A\right)\right)^c & \text{and} & T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c \\
T^{-1}\left(T(A)\right) = A & \text{and} & T\left(T^{-1}(B)\right) = B
\end{array}
$$

Provide counterexamples for each of the false assertions.

(1.3) The set $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is called the ***extended real line***. Write $\mathcal{A}$ for the sigma-field on $\overline{\mathbb{R}}$ generated by $\mathcal{B}(\mathbb{R})$ together with the two singleton sets $\{-\infty\}$ and $\{\infty\}$. Show that $\mathcal{A}$ is also generated by $\mathcal{E} = \{[-\infty, t] : t \in \mathbb{R}\}$.

(1.4) UGMTP Problem 1.1.

(1.5) Suppose $f_1, \ldots, f_n$ are functions in $\mathcal{M}^+$ and μ is a measure for which $\mu f_i < \infty$ for each i. Show that

$$
\sum\nolimits_i \mu f_i - \sum\nolimits_{i<j} \mu(f_i \wedge f_j) \le \mu\left(\max\nolimits_i f_i\right) \le \sum\nolimits_i \mu f_i - \sum\nolimits_{i<j} \mu(f_i \wedge f_j) + \sum\nolimits_{i<j<k} \mu(f_i \wedge f_j \wedge f_k).
$$

Hint: For fixed nonnegative numbers $a_1, \ldots, a_n$ establish inequalities like $\sum_i a_i \le \max_i a_i + \sum_{i<j} a_i \wedge a_j$. For simplicity assume $0 \le a_1 \le a_2 \le \ldots \le a_n$ then consider the coefficients of each a_{n-k}.