Statistics 330/600 **Due: Thursday 12 April**
2007: Sheet 10

*(10.1) Suppose $\{\xi_i : i \in \mathbb{N}\}$ are independent random variables with $\mathbb{P}\{\xi_i = 1\} = 1/2 = \mathbb{P}\{\xi_i = -1\}$, for each i. Suppose $X_0 \equiv 1$ and $X_n = X_0 + \xi_1 + \ldots + \xi_n$. Let $\mathcal{F}_n = \sigma\{X_0, \xi_1, \ldots, \xi_n\}$. Define $\tau = \inf\{n : X_n = 0\}$ and $Z_n = X_{\tau \wedge n}$.

(i) Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale. Does $\{X_n\}$ converge almost surely?

(ii) Show that $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a positive martingale.

(iii) Show that $Z_{n+1} - Z_n \to 0$ almost surely. Hint: What do you know about sequences of real numbers that converge to finite limits?

(iv) From part (iii) and the fact that $|X_{n+1} - X_n| = 1$, deduce that $\tau < \infty$ almost surely.

(v) Show that $\mathbb{P}X_\tau\{\tau < \infty\} \neq \mathbb{P}X_0$. Why does this fact not contradict the Stopping Time Lemma?

(10.2) Suppose W and X are nonnegative random variables with $\|Z\|_p < \infty$ for some $p > 1$. Suppose also that there exists positive constants β and C for which

$$t\mathbb{P}\{W > \beta t\} \leq C\mathbb{P}Z\{W > t\} \qquad \text{for all } t > 0.$$

Show that $\|W\|_p \leq Cp\beta^p\|Z\|_p/(p-1)$. *To make things slightly easier, I will let you assume that* $\|W\|_p < \infty$. *For a truly virtuoso effort, you might also show that the finiteness of* $\|W\|_p$ *actually follows from the finiteness of* $\|Z\|_p$.

*(10.3) (Doob's $\mathcal{L}^p$ maximal inequality) UGMTP Problem 6.9. If you solve HW 10.2, you may appeal to that problem to eliminate some of the steps suggested in the book.

(10.4) Suppose $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a martingale with $\sup_n \mathbb{P}X_n^2 < \infty$. As usual, write X_n as a sum of martingale differences, $X_n = X_0 + \sum_{i \leq n} \xi_i$.

> *For this problem I do not want you to appeal to upcrossing results or convergence results for positive supermartingales. The problem is intended as an explanation of an alternative way to prove convergence.*

(i) For each n and m with $n < m < \infty$, show that $\mathbb{P}|X_m - X_n|^2 = \sum_{i=n+1}^{m} \sigma_i^2$, where $\sigma_i^2 := \mathbb{P}\xi_i^2$.

(ii) For $n < m \leq \infty$, define $\Delta_{n,m} = \sup_{n<i\leq m} |X_i - X_m|$. Use Doob's $\mathcal{L}^2$ maximal inequality (with m finite) to show that $\mathbb{P}\Delta_{n,\infty}^2 \to 0$ as $n \to \infty$.

(iii) Show that $\Delta_{n,\infty} \to 0$ almost surely as $n \to \infty$. Hint: $\Delta_{n,\infty}$ decreases as n increases.

(iv) Show that there exists an X_∞ in $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, \mathbb{P})$ for which $X_n \to X_\infty$ almost surely and $\|X_n - X_\infty\|_2 \to 0$.