Statistics 330/600 **Due: Thursday 19 April**
2007: Sheet 11

*(11.1) Let $\mathbb{P}$ be Lebesgue measure on the Borel sigma-field of $(0,1]$. Let X be a nonnegative, $\mathbb{P}$-integrable random variable defined on $(0,1]$. Define $\mathcal{F}_n$ to be the sigma-field generated by the intervals

$$J_{i,n} := \left(\frac{i-1}{2^n}, \frac{i}{2^n}\right] \qquad \text{for } i = 1, 2, \ldots, 2^n.$$

(i) Show that $X_n := \mathbb{P}_{\mathcal{F}_n} X$ is of the form $\sum_{i=1}^{2^n} x_{i,n}\{\omega \in J_{i,n}\}$ for numbers $x_{i,n}$ that you specify.

(ii) Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a martingale, which converges almost surely to an integrable limit X_∞.

(iii) Use Fatou's lemma to show that $\mathbb{P}X \geq \mathbb{P}X_\infty$.

(iv) Temporarily suppose X is bounded above by a constant. Show that $\mathbb{P}|X_n - X_\infty| \to 0$. Deduce that $\mathbb{P}XJ_{i,k} = \mathbb{P}X_\infty J_{i,k}$ for each $J_{i,k}$. Explain why it follows that $X = X_\infty$ almost surely.

(v) Now remove the temporary assumption of boundedness. For each M in $\mathbb{R}^+$, show that

$$X_n \geq X_{n,M} := \mathbb{P}_{\mathcal{F}_n}(X \wedge M) \to X \wedge M \qquad \text{almost surely.}$$

(vi) Deduce, via (iii) and (v), that $X_\infty = X$ almost surely.

(11.2) For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ for which either $\mathbb{P}G = 0$ or $\mathbb{P}G = 1$ for every $G \in \mathcal{G}$.

(i) For each $Z \in \mathcal{M}^+(\mathcal{G})$, show that there exists a constant $c \in [0, \infty]$ for which $\mathbb{P}\{Z = c\} = 1$.

(ii) For each $X \in \mathcal{M}^+(\mathcal{F})$, show that $\mathbb{P}(X \mid \mathcal{G}) = \mathbb{P}X$ a.e. $[\mathbb{P}]$.

*(11.3) (properties of $d(x, B)$ in a metric space) UGMTP Problem 7.3.