Statistics 330/600 **Due: Thursday 26 April**
2007: Sheet 12

*(12.1) (Fubini and cts a.s.) UGMTP Problem 7.10.

(12.2) Let g be a nonnegative, lower semicontinuous function on a metric space $\mathcal{X}$ (see UGMTP page 172). Suppose $P, P_1, P_2, \dots$ are probability measures on $\mathcal{B}(\mathcal{X})$ for which $P_n \rightsquigarrow P$. Adapt the method from the previous Problem to give an alternative proof of UGMTP Theorem 7.8. That is, use Fubini and the fact that $\liminf_{n\to\infty} P_n G \ge PG$ for each open set G to show that $\liminf_{n\to\infty} P_n g \ge Pg$.

*(12.3) Suppose $P, P_1, P_2, \dots$ are probability measures on $\mathcal{B}(\mathbb{R}^k)$ for which $P_n f \to Pf$ for every Lipschitz function f on $\mathbb{R}^k$ with compact support. Show that $P_n \rightsquigarrow P$. Hint: Show that there exists a Lipschitz function h with compact support for which $0 \le h \le 1$ and $Ph > 1 - \epsilon$. Consider functions fh with $f \in \text{BL}(\mathbb{R}^k)$.

(12.4) Suppose K is an infinitely differentiable function on $\mathbb{R}$ with compact support. Let f be a bounded, $\mathcal{B}(\mathbb{R})$-measurable function. Show that the function $g(t) = \int_{-\infty}^{\infty} f(x)K(x+t)\,dx$ is also infinitely differentiable.

*(12.5) Suppose $P, P_1, P_2, \dots$ are probability measures on $\mathcal{B}(\mathbb{R})$ for which $P_n(-\infty, x] \to P(-\infty, x]$ for every x in some dense subset T of $\mathbb{R}$. Show that $P_n \rightsquigarrow P$ by the following steps.

(i) For each $\epsilon > 0$ show that there exist points $x_1 < x_2 < \ldots < x_k$ in T for which $\max_{i=2}^k |x_i - x_{i-1}| < \epsilon$ and $P(-\infty, x_1] < \epsilon$ and $P(x_k, \infty) < \epsilon$.

(ii) Given $f \in \text{BL}(\mathbb{R})$, define a step function $f_k(x) := \sum_{i=2}^k f(x_i)\{x_{i-1} < x \le x_i\}$. Show that $P_n f_k \to P f_k$ for each k and that both $P|f - f_k|$ and $\limsup_n P_n|f - f_k|$ are suitably small.

(iii) Complete the proof.

(12.6) For each n, suppose X_n and Y_n are independent, real random variables. Suppose also that $X_n \rightsquigarrow P$ and $Y_n \rightsquigarrow Q$, for some probability measures P and Q on $\mathcal{B}(\mathbb{R})$. Without using any Fourier arguments, show that (X_n, Y_n) converges in distribution, as a random element of $\mathbb{R}^2$, to $P \otimes Q$. Hint: Adapt the argument from the previous Problem to show that $\mathbb{R}^2$ can be partitioned into countably many rectangles of the form $J = (a, b] \times (c, d]$, each with diameter less than ϵ, and for each of which $\mathbb{P}\{(X_n, Y_n) \in J\} \to P \otimes QJ$. Then consider $\mathbb{P}f(X_n, Y_n)$ for a Lipschitz function f with compact support.