Statistics 330/600 **Due: Thursday 8 February**
2007: Sheet 3

Please attempt at least three problems.

(3.1) For each θ in [0, 1] let $X_{n,\theta}$ be a random variable with $\mathbb{P}\{X_{n,\theta} = k\} = \binom{n}{k}\theta^k(1-\theta)^{n-k}$ for $k = 0, 1, \ldots, n$. (That is, $X_{n,\theta}$ has a Binomial(n, θ) distribution.) You may assume these elementary facts: $\mathbb{P}X_{n,\theta} = n\theta$ and $\mathbb{P}(X_{n,\theta} - n\theta)^2 = n\theta(1-\theta)$. Let f be a continuous function defined on [0, 1].

(i) Show that $p_n(\theta) = \mathbb{P}f(X_{n,\theta}/n)$ is a polynomial in θ.

(ii) Suppose $\sup_{0\le x\le 1} |f(x)| = M < \infty$. For a fixed $\epsilon > 0$, invoke (uniform) continuity to find a $\delta > 0$ such that $|f(s) - f(t)| \le \epsilon$ whenever $|s - t| \le \delta$, for all s, t in [0, 1]. Explain why

$$|f(x/n) - f(\theta)| \le \epsilon + 2M\{\,|(x/n) - \theta| > \delta\} \le \epsilon + 2M|(x/n) - \theta|^2/\delta^2.$$

(iii) Deduce that $\sup_{0\le\theta\le 1} |p_n(\theta) - f(\theta)| < 2\epsilon$ for n large enough. That is, deduce that $f(\cdot)$ can be uniformly approximated by polynomials over the range [0, 1], a result known as the ***Weierstrass approximation theorem***.

(3.2) Suppose a sequence of random variables $\{X_n : n \in \mathbb{N}\}$ converges in probability to zero, that is, $\mathbb{P}\{|X_n| > \epsilon\} \to 0$ as $n \to \infty$ for each $\epsilon > 0$. Show that there is an increasing sequence of positive integers $\{n(k) : k \in \mathbb{N}\}$ for which $\sum_k \mathbb{P}\{|X_{n(k)}| > 1/k\} < \infty$. Deduce that $X_{n(k)} \to X$ almost surely.

(3.3) Suppose $f \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Show that $\sum_{i=1}^n \mu|f|\{|f| > i\}/n \to 0$ as $n \to \infty$.

(3.4) (converse BC) UGMTP Problem 2.1 or UGMTP Problem 2.2.

(3.5) For each i in some index set $\mathcal{I}$ suppose $\mathcal{Z}_i$ is a set equipped with a sigma-field $\mathcal{C}_i$. Suppose $\mathcal{Y}$ is another set and we are given functions $h_i : \mathcal{Y} \to \mathcal{Z}_i$ for each i. Let $\mathcal{H} = \{h_i : i \in \mathcal{I}\}$.

(i) Let $\mathcal{E} := \{h_i^{-1}(C_i) : C_i \in \mathcal{C}_i\ , i \in \mathcal{I}\}$. Show that $\sigma(\mathcal{E})$ is the smallest sigma-field $\mathcal{B}$ on $\mathcal{Y}$ for which each h_i is $\mathcal{B}\backslash\mathcal{C}_i$-measurable. The sigma-field $\sigma(\mathcal{E})$ is usually denoted by $\sigma(\mathcal{H})$.

(ii) Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and that $T : \mathcal{X} \to \mathcal{Y}$. Show that T is $\mathcal{A}\backslash\sigma(\mathcal{H})$-measurable if and only if $h_i \circ T$ is $\mathcal{A}\backslash\mathcal{C}_i$-measurable for all $i \in \mathcal{I}$.