Statistics 330/600 **Due: Thursday 22 February**
2007: Sheet 5

Please attempt at least the starred problems and at least one of the unstarred problems.

*(5.1) The following facts are sometimes useful if you need to establish product-measurability of sets or functions.

(i) Show that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$. Hint: For the inclusion $\subseteq$, show that every open subset of $\mathbb{R}^2$ can be written as a countable union of sets of the form $(a, b) \times (c, d)$.

(ii) Show that $\{(x, y) \in \mathbb{R}^2 : x = y\} \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$.

(iii) Show that every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$-measurable.

*(5.2) (Compare with UGMTP Problem 2.19.) For a measure space $(\mathcal{X}, \mathcal{A}, \mu)$, suppose $\{f_n : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^p(\mu)$, for some fixed $p > 1$, that is, $\|f_n - f_m\|_p \to 0$ as $\min(m, n) \to \infty$. Show that there exists a real-valued function f in $\mathcal{L}^p(\mu)$ for which $\|f_n - f\|_p \to 0$ as $n \to \infty$, by the following steps.

(i) Show that there is no loss of generality in supposing $f_n \geq 0$ for all n. Hint: Note that $|a^+ - b^+| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

(ii) For all $\alpha, \beta \in \mathbb{R}^+$, show that

$$|\alpha - \beta|^p \leq |\alpha^p - \beta^p| \leq p|\alpha - \beta| \left(\alpha^{p-1} + \beta^{p-1}\right).$$

Hint: Reduce to the case $\alpha > \beta = 1$.

(iii) Show that $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$.

(iv) Show that $\{f_n^p : n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^1(\mu)$. Hint: Hölder.

(v) Show that there exists a function g in $\mathcal{L}^p(\mu)$ with $g \geq 0$ and $\mu|f_n^p - g^p| \to 0$.

(vi) Show that $\|f_n - g\|_p \to 0$.

(vii) Anything more to do?

(5.3) Suppose X and Y are independent random variables for which $\mathbb{P}\{X + Y = 1\} = 1$. Show that there exists some constant C for which $X = C$ almost surely.

(5.4) Let P be a probability measure on $\mathcal{B}(\mathbb{R})$ with distribution function $F(\cdot)$ and corresponding quantile function $q(\cdot)$. Let $m_0 = q(1/2)$. Show that $P[m_0, \infty) \geq 1/2$ and $P(-\infty, m_0] \geq 1/2$. Explain why m_0 is the smallest value for which both these inequalities hold. *The value m_0 is called a median for P.*

(5.5) Suppose $Z = X + Y$, with X and Y independent random variables. Let m be a median for the distribution of Y. Show that $\mathbb{P}\{X \geq x\} \leq 2\mathbb{P}\{Z \geq x + m\}$ for each real x.