Statistics 330/600 **Due: Thursday 8 March**
2007: Sheet 7

Please attempt the starred problems and at least two of the unstarred problems.

*(7.1) UGMTP Problem 3.12, part (i). You could also attempt part (ii) if you read the first two pages of Section 3.3.

*(7.2) Let $\mathcal{E}$ be a field on a set $\mathcal{X}$. That is, $\emptyset \in \mathcal{E}$ and $\mathcal{E}$ is stable under the formation of complements, finite unions, and finite intersections. Suppose μ and ν are finite measures on $\sigma(\mathcal{E})$. Define $\lambda = \mu + \nu$.

(i) From UGMTP Example 2.5, for each $B \in \sigma(\mathcal{E})$ and each $\epsilon > 0$ there exists an increasing sequence of sets $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{E}$ for which $B \subseteq \cup_{n \in \mathbb{N}} E_n$ and $\lambda\left(\cup_{n\in\mathbb{N}} E_n\right) < \epsilon + \lambda B$. Deduce that there exists a set $E \in \mathcal{E}$ for which $\lambda|B - E| < \epsilon$.

(ii) Suppose that for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\nu E < \epsilon$ for every set $E \in \mathcal{E}$ with $\mu E < \delta$. Show that $\nu \ll \mu$, as measures on $\sigma(\mathcal{E})$. Hint: If $\mu B = 0$ find an $E \in \mathcal{E}$ with $\lambda|B - E| < \delta \wedge \epsilon$.

(iii) Let μ be Lebesgue measure on $\mathcal{B}(0, 1]$ and ν be a finite measure on $\mathcal{B}(0, 1]$ with distribution function F. Suppose that for each $\epsilon > 0$ there exists a $\delta > 0$ with the property that $\sum_i \left(F(b_i) - F(a_i)\right) < \epsilon$ for every finite set of points $0 \le a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n \le 1$ for which $\sum_i (b_i - a_i) < \delta$. Show that $\nu \ll \mu$.

(iv) For the ν and F from the previous part, show that there exists a nonnegative, Lebesgue-integrable function f for which $F(x) = \int_0^x f(t)\,dt$ for each x in $(0, 1]$.

(7.3) UGMTP Problem 3.16.

(7.4) UGMTP Problem 3.17.

(7.5) Let ν be a finite measure (on a sigma-field $\mathcal{A}$) for which $\nu = \nu_1 + \lambda_1 = \nu_2 + \lambda_2$, where each ν_i is dominated by a fixed sigma-finite measure μ and each λ_i is singular with respect to μ. Show that $\nu_1 = \nu_2$ and $\lambda_1 = \lambda_2$. Hint: Show that there exists a set $A \in \mathcal{A}$ for which $\mu A^c = 0$ and $\lambda_1 A + \lambda_2 A = 0$. Consider $\nu(f A)$ for $f \in \mathcal{M}^+$.