Statistics 330/600 **Due: Thursday 5 April**
2007: Sheet 9

*(9.1) Suppose $\{\mathbb{P}_t : t \in \mathcal{T}\}$ and $\{\widetilde{\mathbb{P}}_t : t \in \mathcal{T}\}$ are two candidates for the conditional probability distribution described in UGMTP Definition 5.4. Suppose the sigma-field $\mathcal{F}$ is countably generated, that is, $\mathcal{F} = \sigma(\mathcal{E})$ for some countable collection $\mathcal{E}$ of sets.

(i) Show that there is no loss of generality in assuming that $\mathcal{E}$ is stable under the formation of finite intersections.

(ii) For each $E \in \mathcal{E}$, show that $\mathbb{P}_t E = \widetilde{\mathbb{P}}_t E$ a.e $[\mathbb{Q}]$. Hint: Consider $\mathbb{P}\{\omega \in E, T\omega \in B\}$ for various $B \in \mathcal{B}$.

(iii) Show that there exists a $\mathbb{Q}$-negligible set $\mathcal{N}$ such that $\mathbb{P}_t = \widetilde{\mathbb{P}}_t$, as measures on $\mathcal{F}$, for all $t \in \mathcal{N}^c$.

*(9.2) In class I considered a probability measure $\mathbb{P} = P \otimes P$ on $\mathcal{B}(\mathbb{R}^2)$, where P is a nonatomic measure with distribution function F. For the map $T(x, y) = \max(x, y)$, with distribution Q, I asserted that the conditional distributions take the form $\mathbb{P}_t = \frac{1}{2}\mu_t \otimes \delta_t + \frac{1}{2}\delta_t \otimes \mu_t$, where δ_t denotes the probability measure concentrated at the single point t and μ_t denotes the probability measure with density $\{x \le t\}/F(t)$ with respect to P, provided $F(t) > 0$.

(i) Show that $Q\{t : F(t) = 0\} = 0$.

(ii) For some arbitrary probability measure ν on $\mathcal{B}(\mathbb{R}^2)$, define $\mathbb{P}_t = \nu$ when $F(t) = 0$. Show that $\{\mathbb{P}_t : t \in \mathbb{R}\}$ is (one possible choice for) the conditional distributions.

For the next problem you may assume the following result. Suppose that $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-sigma-field of $\mathcal{F}$. If g is a $\mathcal{G}$-measurable, real-valued random variable on Ω for which $gX \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathbb{P}_\mathcal{G}(gX) = g\mathbb{P}_\mathcal{G}X$ almost surely.

*(9.3) Suppose $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$. Show that $\text{var}(X) = \text{var}(\mathbb{P}_\mathcal{G}X) + \mathbb{P}(\text{var}_\mathcal{G}X)$, where $\text{var}_\mathcal{G}(X) = \mathbb{P}_\mathcal{G}\left((X - \mathbb{P}_\mathcal{G}X)^2\right)$.