Statistics 330/600 **Due: Thursday 24 January**
2008: Sheet 1

Please attempt at least the starred problems.

*(1.1) Please send to me (david.pollard@yale.edu) an email stating whether you have taken (or are in the process of taking) real analysis, measure theory, or any probability course.

*(1.2) Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For $B \subseteq \mathcal{Y}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. For $A \subseteq \mathcal{X}$ define $T(A) := \{T(x) : x \in A\}$. Some of the following assertions are true in general and some are false.

$$T\left(\cup_i A_i\right) = \cup_i T(A_i) \quad \text{and} \quad T^{-1}\left(\cup_i B_i\right) = \cup_i T^{-1}(B_i)$$
$$T\left(\cap_i A_i\right) = \cap_i T(A_i) \quad \text{and} \quad T^{-1}\left(\cap_i B_i\right) = \cap_i T^{-1}(B_i)$$
$$T\left(A^c\right) = \left(T\left(A\right)\right)^c \quad \text{and} \quad T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c$$
$$T^{-1}\left(T(A)\right) = A \quad \text{and} \quad T\left(T^{-1}(B)\right) = B$$

Provide counterexamples for each of the false assertions.

*(1.3) Each x in $[0, 1)$ has a unique binary expansion $x = \sum_{k\in\mathbb{N}} x_k/2^k$, with $x_k \in \{0, 1\}$, provided we exclude the possibility that $x_k = 1$ for all k large enough. (So, for example, we choose for $1/2$ the expansion with $x_1 = 1$ and $x_k = 0$ for $k \geq 2$ rather than the expansion with $x_1 = 0$ and $x_k = 1$ for all $k \geq 2$.) Define a map T from $[0, 1)$ into $\Omega = \{0, 1\}^{\mathbb{N}}$ by $Tx = (x_1, x_2, \ldots)$.

For each k in $\mathbb{N}$ and each subset B of $\{0, 1\}^k$ define the "cylinder set with base B" to be

$$\widetilde{B} = \{\omega \in \Omega : (\omega_1, \ldots, \omega_k) \in B\}$$

Write $\mathcal{E}$ for the collection of all such cylinder subsets of Ω. (That is, $\mathcal{E}$ consists of all $\widetilde{B}$ as k ranges over $\mathbb{N}$ and B ranges over all subsets of $\{0, 1\}^k$.) Define $\mathcal{F} = \sigma(\mathcal{E})$.

(i) Show that $\{x \in [0, 1) : x_k = 1\}$ is a finite union of disjoint intervals.

(ii) Show that $T^{-1}\widetilde{B} \in \mathcal{B}[0, 1)$ for each cylinder set $\widetilde{B}$. Hint: Consider first the case where B consists of a single point in $\{0, 1\}^k$. Don't get carried away with notation.

(iii) Deduce that T is $\mathcal{B}[0, 1)\backslash\mathcal{F}$-measurable.

(iv) Let $\mathfrak{m}$ denote Lebesgue measure on $\mathcal{B}[0, 1)$. Define $\mathbb{P}$ on $\mathcal{F}$ by $\mathbb{P}F = \mathfrak{m}(T^{-1}F)$. Show that $\mathbb{P}$ is a probability measure.

(v) For each cylinder set $\widetilde{B}$ with base B in $\{0, 1\}^k$, show that $\mathbb{P}\widetilde{B} = (\text{number of points in } B)/2^k$.

(1.4) The set $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is called the ***extended real line***. Write $\mathcal{A}$ for the sigma-field on $\overline{\mathbb{R}}$ generated by $\mathcal{B}(\mathbb{R})$ together with the two singleton sets $\{-\infty\}$ and $\{\infty\}$. Show that $\mathcal{A}$ is also generated by $\mathcal{E} = \{\,[-\infty, t] : t \in \mathbb{R}\}$.