Statistics 330/600 **Due: Thursday 17 April**
2008: Sheet 10

Throughout the sheet let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a fixed probability space.*

*(10.1) Let τ_1 and τ_2 be stopping times for a filtration $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$.

(i) Show that $\mathcal{F}_{\tau_1 \wedge \tau_2} = \mathcal{F}_{\tau_1} \cap \mathcal{F}_{\tau_2}$.

(ii) Show that $\mathcal{F}_{\tau_1 \vee \tau_2} = \sigma\big(\mathcal{F}_{\tau_1} \cup \mathcal{F}_{\tau_2}\big)$. Hint: Show that each set of the form $F\{_1 \vee \tau_2 = i$ with $F \in \mathcal{F}_i$ can be written as a union of two sets, one in $\mathcal{F}_{\tau_1}$, the other in $\mathcal{F}_{\tau_2}$.

(10.2) Let $\{(Z_n, \mathcal{F}_n) : n \in\}$ be a submartingale and τ be a stopping time. Show that $\{(Z_{\tau\wedge n}, \mathcal{F}_n) : n \in\}$ is also a submartingale. Hint: For F in $\mathcal{F}_{n-1}$, consider separately the contributions to $\mathbb{P}Z_{n\wedge\tau}F$ and $\mathbb{P}Z_{(n-1)\wedge\tau}F$ from the regions $\{\tau \le n-1\}$ and $\{\tau \ge n\}$.

(10.3) Suppose $\{Z_n : n \in \mathbb{N}_0\}$ are random variables on Ω. Define a filtration by $\mathcal{F}_n = \sigma\{Z_i : i \le n\}$ for each $n \in \mathbb{N}_0$. Define $\mathcal{F}_\infty = \sigma\{Z_i : i \in \mathbb{N}_0\} = \sigma\big(\cup_{n\in\mathbb{N}_0} \mathcal{F}_n\big)$. Suppose τ is a stopping time for the filtration. Define $X_i = Z_{\tau\wedge i}$ and $\mathcal{G}_n = \sigma\{X_i : i \le n\}$ and $\mathcal{G}_\infty = \sigma\{X_i : i \in \mathbb{N}_0\}$. Show that $\mathcal{F}_\tau = \mathcal{G}$ by the following steps.

As a convenient abbreviation, write W_n for the random vector $(Z_0, Z_1, \dots, Z_n)$ and W_∞ for $\mathbb{R}^{\mathbb{N}_0}$ random element $(Z_0, Z_1, \dots)$. Write Y_n and Y_∞ for the analogous variables defined from the X_i's.

(i) Show that X_i is $\mathcal{F}_\tau$-measurable. Hint: You may assume the result proved at the bottom of UGMTP page 143. Deduce that $\mathcal{G}_\infty \subseteq \mathcal{F}_\tau$.

(ii) Explain why there exist sets $A_i \in \mathcal{B}(\mathbb{R}^{i+1})$ for which $\{\tau = i\} = \{W_i \in A_i\}$ for each $i \in \mathbb{N}_0$.

(iii) Use the result from the next Problem on this Sheet to explain why there exists a set $A_\infty \in \mathcal{B}(\mathbb{R}^{\mathbb{N}_0})$ such that $\{\tau = \infty\} = \{W_\infty \in A_\infty\}$.

(iv) Suppose $F \in \mathcal{F}_\tau$. For each k in $\mathbb{N}_0$, explain why $F\{\tau = k\} = \{W_k \in B_k\}$ for some $B_k \in \mathcal{B}(\mathbb{R}^{k+1})$.

(v) For each k in $\mathbb{N}_0$, show that $\{\tau \ge k\} \in \mathcal{G}_{k-1}$. Hint for $k = 2$: First show that $\{\tau \ge 2\}$ can also be written as $\{W_0 \notin A_0, W_1 \notin A_2\}\{\tau \ge 1\}$.

(vi) For each k in $\mathbb{N}_0$, show that

$$F\{\tau = k\} = \{W_k \in B_k,\ \tau \ge k\} = \{Y_k \in B_k,\ \tau \ge k\} \in \mathcal{G}_k.$$

(vii) Show that $F\{\tau = \infty\} \in \mathcal{G}_\infty$.

(viii) Deduce that $F \in \mathcal{G}_\infty$. Conclude that $\mathcal{F}_\tau \subseteq \mathcal{G}_\infty$.

(10.4) Suppose $\psi_i : \mathcal{Y} \to \mathcal{Z}_i$ and $\mathcal{Z}_i$ is equipped with a sigma-field $\mathcal{C}_i$, for each i in some index set $\mathcal{I}$. Define $\mathcal{B}$ as the smallest sigma-field on $\mathcal{Y}$ for which each ψ_i is $\mathcal{B}\backslash\mathcal{C}_i$-measurable.

(i) Show that $\mathcal{B}$ is generated by the collection of sets $\cup_{i\in\mathcal{I}} \mathcal{E}_i$, where $\mathcal{E}_i := \{\psi_i^{-1}(C) : C \in \mathcal{C}_i\}$.

(ii) Suppose $T : \Omega \to \mathcal{Y}$ and that $\mathcal{F}$ is a sigma-field on Ω. Show that T is $\mathcal{F}\backslash\mathcal{B}$-measurable if and only if $\psi_i \circ T$ is $\mathcal{F}\backslash\mathcal{C}_i$-measurable for each $i \in \mathcal{I}$.

(iii) Specialize to the case where $\mathcal{I} = \mathbb{N}$ and $\mathcal{Y} = \mathbb{R}^{\mathbb{N}}$, with ψ_i as the ith coordinate map: if $x = (x_i : i \in \mathbb{N}) \in \mathbb{R}^{\mathbb{N}}$ then $\psi_i(x) = x_i$. Suppose $T(\omega) = (X_1(\omega), X_2(\omega), \dots)$ for some sequence of $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable real random variables $X_1, X_2, \dots$. Show that T is $\mathcal{F}\backslash\mathcal{B}$-measurable.

(iv) Show that $\sigma(T) = \{T^{-1}(B) : B \in \mathcal{B}\}$.

(v) Show that $\sigma(T) = \mathcal{F}_\infty$, the smallest sigma-field on Ω for which X_i is $\mathcal{F}_\infty\backslash\mathcal{B}(\mathbb{R})$-measurable for each $i \in \mathbb{N}$.

(vi) Show that the elements of $\mathcal{M}^+(\Omega, \mathcal{F}_\infty)$ are precisely those functions of the form $g(T(\omega))$ for some $g \in \mathcal{M}^+(\mathbb{R}^{\mathbb{N}}, \mathcal{B})$. Hint: Start with the bounded random variables.