Statistics 330/600 **Due: Thursday 14 February ♡**
2008: Sheet 4

Please attempt at least the starred problems.

*(4.1) Suppose $\mathcal{E}_i$ is a collection of subsets of $\mathcal{X}_i$ with $\mathcal{X}_i \in \mathcal{E}_i$, for $i = 1, 2$. Show that $\sigma\left(\mathcal{E}_1 \times \mathcal{E}_2\right) = \sigma(\mathcal{E}_1) \otimes \sigma(\mathcal{E}_2)$, as sigma-fields on $\mathcal{X}_1 \times \mathcal{X}_2$.

*(4.2) UGMTP Problem 4.16, parts (i) and (ii) only. Hint: Consider the sum of the indicator functions of the sets $\{(x,t) \in \mathbb{R}^2 : x \le t\}$ and $\{(x,t) \in \mathbb{R}^2 : x \ge t\}$. You may use the fact that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R})$ without proving it—see UGMTP Problem 4.6.

*(4.3) Suppose $\mathcal{A}$ is a sigma-field on a set $\mathcal{X}$ and $\mathcal{B}$ is a sigma-field on a set $\mathcal{Y}$. Suppose $\{\mu_i : i \in \mathbb{N}\}$ is a sequence of measures on $\mathcal{A}$.

(i) Show that the functional $\mu : \mathcal{M}^+(\mathcal{X}, \mathcal{A}) \to [0, \infty]$ defined by $\mu(f) = \sum_{i \in \mathbb{N}} \mu_i f$ corresponds to a countably additive measure on $\mathcal{A}$.

(ii) Suppose μ can be represented as a countable sum of measures as in (i), with each μ_i a finite measure (that is, $\mu_i \mathcal{X} < \infty$ for each i). Must μ be sigma-finite?

(iii) Suppose μ is the measure defined in (ii) and ν is a measure defined similarly from a countable set of finite measures $\{\nu_j : j \in \mathbb{N}\}$ on $\mathcal{B}$. Show that the Tonelli Theorem also works for μ and ν. That is, show that the assumptions of sigma-finiteness in UGMTP Theorem 4.25 can be relaxed.