Throughout the sheet let \((\Omega, \mathcal{F}, \mathbb{P})\) be a fixed probability space.

*(5.1)* Let \(X\) be a random variable on \(\Omega\). Show that \(\{(\omega, s) \in \Omega \times \mathbb{R} : X(\omega) > s\} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})\).

*(5.2)* Let \(X_1, X_2, \ldots\), be independent random variables with \(\mathbb{P}X_i = 0\) and \(\mathbb{P}X_i^6 \leq C\) for each \(i\), where \(C\) is a finite constant. Show that \(\mathbb{P}(X_1 + \ldots + X_n)^6 \leq C_0 n^3\) for each \(n\), where \(C_0\) is a constant that depends only on \(C\).

*(5.3)* Let \(\mathcal{X}\) be a set equipped with a *countably generated* sigma-field \(\mathcal{A}\). That is, \(\mathcal{A} = \sigma(\mathcal{E})\) for some countable collection of sets \(\mathcal{E} = \{E_i : i \in \mathbb{N}\}\). Suppose also that \(\{x\} \in \mathcal{A}\) for each \(x\) in \(\mathcal{X}\). [Note that there is no loss of generality in assuming \(\mathcal{E}\) to be stable under complements, for we could replace \(\mathcal{E}\) by \(\mathcal{E} \cup \{E^c : E \in \mathcal{E}\}\).]

Suppose \(X : \Omega \to \mathcal{X}\) is an \(\mathcal{F} \setminus \mathcal{A}\)-measurable map that is independent of itself. Show that there exists some \(x_0 \in \mathcal{X}\) for which \(X = x_0\) almost surely \([\mathbb{P}]\) by the following steps.

(i) For each pair of points \(x_1 \neq x_2\) in \(\mathcal{X}\), show that there must exists some \(E_i\) for which \(x_1 \in E_i, x_2 \in E_i^c\).

Hint: If the two points could not be separated in this way, consider

\[
\{A \in \mathcal{A} : \text{either } \{x_1, x_2\} \subseteq A \text{ or } \{x_1, x_2\} \subseteq A^c\}
\]

(ii) For each \(i\), show that \(\mathbb{P}\{X \in E_i\}\) is either zero or one. Hint: Consider \(\mathbb{P}\{X \in E_i, X \in E_i^c\}\).

(iii) Define \(A_0\) to be the intersection of all those \(E_i\) for which \(\mathbb{P}\{X \in E_i\} = 1\). Show that \(\mathbb{P}\{X \in A_0\} = 1\).

(iv) Show that \(A_0\) is a singleton set. Hint: Use part (i).

*(5.4)* (Rio’s inequality) Let \(\mathcal{A}\) and \(\mathcal{B}\) be two sub-sigma-fields of \(\mathcal{F}\). Define

\[
\alpha := \sup\{|\mathbb{P}(AB) - (\mathbb{P}A)(\mathbb{P}B)| : A \in \mathcal{A}, B \in \mathcal{B}\}.
\]

Suppose \(X\) is nonnegative and \(\mathcal{A}\)-measurable, and \(Y\) is nonnegative and \(\mathcal{B}\)-measurable. Write \(q_X\) for the quantile function corresponding to the distribution of \(X\) and \(q_Y\) be the analogous quantile function for \(Y\). Show that

\[
|\text{cov}(X, Y)| \leq \int_0^\alpha q_X(u)q_Y(u) \, du.
\]

Hint: First show that \(|\text{cov}(X, Y)|\) is smaller than some double integral of \(\min(\alpha, \mathbb{P}(X > x), \mathbb{P}(Y > y))\).