Statistics 330/600 **Due: Thursday 21 February**
2008: Sheet 5

Throughout the sheet let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a fixed probability space.*

*(5.1) Let X be a random variable on Ω. Show that $\{(\omega, s) \in \Omega \times \mathbb{R} : X(\omega) > s\} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

*(5.2) Let $X_1, X_2 \dots,$ be independent random variables with $\mathbb{P}X_i = 0$ and $\mathbb{P}X_i^6 \le C$ for each i, where C is a finite constant. Show that $\mathbb{P}\big(X_1 + \dots + X_n\big)^6 \le C_0 n^3$ for each n, where C_0 is a constant that depends only on C.

*(5.3) Let $\mathcal{X}$ be a set equipped with a ***countably generated*** sigma-field $\mathcal{A}$. That is, $\mathcal{A} = \sigma(\mathcal{E})$ for some countable collection of sets $\mathcal{E} = \{E_i : i \in \mathbb{N}\}$. Suppose also that $\{x\} \in \mathcal{A}$ for each x in $\mathcal{X}$. [Note that there is no loss of generality in assuming $\mathcal{E}$ to be stable under complements, for we could replace $\mathcal{E}$ by $\mathcal{E} \cup \{E^c : E \in \mathcal{E}\}$.]

Suppose $X : \Omega \to \mathcal{X}$ is an $\mathcal{F}\backslash\mathcal{A}$-measurable map that is independent of itself. Show that there exists some $x_0 \in \mathcal{X}$ for which $X = x_0$ almost surely [$\mathbb{P}$] by the following steps.

(i) For each pair of points $x_1 \ne x_2$ in $\mathcal{X}$, show that there must exists some E_i for which $x_1 \in E_i$, $x_2 \in E_i^c$. Hint: If the two points could not be separated in this way, consider

$$\{A \in \mathcal{A} : \text{either } \{x_1, x_2\} \subseteq A \text{ or } \{x_1, x_2\} \subseteq A^c\}$$

(ii) For each i, show that $\mathbb{P}\{X \in E_i\}$ is either zero or one. Hint: Consider $\mathbb{P}\{X \in E_i, X \in E_i^c\}$.

(iii) Define A_0 to be the intersection of all those E_i for which $\mathbb{P}\{X \in E_i\} = 1$. Show that $\mathbb{P}\{X \in A_0\} = 1$.

(iv) Show that A_0 is a singleton set. Hint: Use part (i).

(5.4) (Rio's inequality) Let $\mathcal{A}$ and $\mathcal{B}$ be two sub-sigma-fields of $\mathcal{F}$. Define

$$\alpha := \sup\{|\mathbb{P}(AB) - (\mathbb{P}A)(\mathbb{P}B)| : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Suppose X is nonnegative and $\mathcal{A}$-measurable, and Y is nonnegative and $\mathcal{B}$-measurable. Write q_X for the quantile function corresponding to the distribution of X and q_Y be the analogous quantile function for Y. Show that

$$|\text{cov}(X, Y)| \le \int_0^\alpha q_X(u) q_Y(u)\, du.$$

Hint: First show that $|\text{cov}(X, Y)|$ is smaller than some double integral of $\min\big(\alpha, \mathbb{P}\{X > x\}, \mathbb{P}\{Y > y\}\big)$.