Statistics 330/600 **Due: Thursday 28 February**
2008: Sheet 6

Throughout the sheet let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a fixed probability space.*

*(6.1) Let $X_1, \ldots, X_n$ be independent random k-vectors (that is, $\mathcal{F}\backslash\mathcal{B}(\mathbb{R}^k)$ measurable maps into R^k), each with a symmetric distribution (that is, X_i has the same distribution as $-X_i$). Define $S_i = X_1 + \ldots + X_i$. For a fixed $\lambda > 0$, show that $\mathbb{P}\{\max_{i\le n}|S_i| > \lambda\} \le 2\mathbb{P}\{|S_n| > \lambda\}$, where $|\cdot|$ denotes the usual Euclidean length. Hint: Define $A_i = \{|S_i| > \lambda,\ |S_j| \le \lambda \text{ for all } j < i\}$ and $T_i = S_n - S_i$. Show that

$$\mathbb{P}A_i \le \mathbb{P}A_i\big(\{|S_i + T_i| > \lambda\} + \{|S_i - T_i| > \lambda\}\big) = 2\mathbb{P}A_i\{|S_n| > \lambda\}.$$

(6.2) Theorem 4.4 of UGMTP established a SLLN for independent random variables $X_1, X_2, \ldots$ with $\mathbb{P}X_i = 0$ and $\sup_i \mathbb{P}X_i^4 < \infty$. Show the the SLLN also holds if we relax the fourth moment assumption to $\mathbb{P}X_i^4 = O(i^\alpha)$ for some $\alpha < 1$.

*(6.3) Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, whichis equipped with a σ-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which T is $\sigma(T)\backslash\mathcal{B}$-measurable. Show that to each f in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a $\mathcal{B}\backslash\mathcal{B}[0,\infty]$-measurable function g from $\mathcal{Y}$ into $[0,\infty]$ such that $f(x) = g(T(x))$, for all x in $\mathcal{X}$, by following these steps.

(i) Consider the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(ii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to f. Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property.

(iii) In part (ii), why can't we assume that $\lim g_n(y)$ exists for each y?

*(6.4) Let $\mathcal{A}$ be a sigma-field on a set $\mathcal{X}$ and $\mathcal{B}$ be a sigma-field on a set $\mathcal{Y}$. Let T be an $\mathcal{A}\backslash\mathcal{B}$-measurable map from $\mathcal{X}$ into $\mathcal{Y}$. Suppose ν and μ are finite measures on $\mathcal{A}$, with $\nu \ll \mu$. (That is, if $A \in \mathcal{A}$ and $\mu A = 0$ then $\nu A = 0$.)

(i) Define μ_T to be the image of μ under T and ν_T to be the image of ν under T Show that $\nu_T \ll \mu_T$. Write g for the density $d(T\nu)/d(T\mu)$.

(ii) Let ν_0 and μ_0 denote the restrictions of the two measures to the sigma-field $\sigma(T)$. Show that $g \circ T$ is a version of the density $d\nu_0/d\mu_0$. Hint: Look at the previous Problem.

Correction to Homework Problem 5.4: The inequality should be

$$|\mathrm{cov}(X, Y)| \le \int_0^\alpha q_X(1-u)q_Y(1-u)\,du.$$