Statistics 330/600 **Due: Thursday 6 March**
2008: Sheet 7

(7.1) *UGMTP Problem 3.13.*

(7.2) *Let P and Q be probability measures defined on the same sigma-field $\mathcal{F}$. A randomized test is a measurable function f with $0 \le f \le 1$. (For observation ω, the value $f(\omega)$ gives the probability of rejecting the "hypothesis" P.) For fixed constants $\alpha > 0$ and $\beta > 0$, find the test that minimizes $\alpha P f + (1 - f)$.*

(7.3) *Read UGMTP page 61 for the definition of Hellinger distance. Suppose P_1 and Q_1 are probability measures on $(\mathcal{X}_1, \mathcal{A}_1)$ and P_2 and Q_2 are probability measures on $(\mathcal{X}_2, \mathcal{A}_2)$. Show that*

$$H^2(P_1 \otimes P_2, Q_1 \otimes Q_2) = 2 - 2\prod\nolimits_{i=1}^{2}\left(1 - \tfrac{1}{2}H^2(P_i, Q_i)\right) \le H^2(P_1, Q_1) + H^2(P_2, Q_2).$$

Hint: Work with densities with respect to some product measure $\lambda_1 \otimes \lambda_2$.

(7.4) *Let $\{\mathbb{P}_\theta : \theta \in \Theta\}$ be a set of probability measures on $(\Omega, \mathcal{F})$, where the index set Θ is a subset of $\mathbb{R}$. Suppose T is an $\mathcal{F}\backslash\mathcal{B}(\mathbb{R})$-measurable function with the property that: for some fixed $\epsilon > 0$ and $\eta > 0$,*

$$\mathbb{P}_\theta\{\omega : |T(\omega) - \theta| > \epsilon\} \le \eta \qquad \text{for each } \theta \in \Theta.$$

- *(i) Suppose $\theta_0, \theta_1 \in \Theta$ are such that $|\theta_0 - \theta_1| > 2\epsilon$. Show that the total variation distance $d_{\text{TV}}(\mathbb{P}_{\theta_0}, \mathbb{P}_{\theta_1})$ is at least $1 - 2\eta$. Hint: Consider $\mathbb{P}_{\theta_i} A$ where $A = \{|T - \theta_0| \le \epsilon\}$.*
- *(ii) Suppose Θ_0 and Θ_1 are two subsets of Θ for which $|\theta_0 - \theta_1| > 2\epsilon$ for all $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$. Show that $d_{\text{TV}}(\mathbb{P}, \mathbb{Q}) \ge 1 - 2\eta$ for each probability measure $\mathbb{P}$ expressible as a convex combination of measures from $\{\mathbb{P}_\theta : \theta \in \Theta_0\}$ and each probability measure $\mathbb{Q}$ expressible as a convex combination of measures from $\{\mathbb{P}_\theta : \theta \in \Theta_1\}$.*