Statistics 330/600 **Due: Thursday 3 April**
2008: Sheet 8

*(8.1) *[conditional Jensen] UGMTP Problem 5.13.*

*(8.2) *[See the second Remark on UGMTP page 127 for the significance of this Problem.] Let $\mathbb{P}$ be Lebesgue measure on $\mathcal{F}$, the Borel sigma-field of $[0,1]$. Let $\mathcal{G}$ denote the sigma-field generated by all the singletons in $[0,1]$.*

- *(i) Show that each set in $\mathcal{G}$ is either countable or its complement is countable, and hence it has probability either zero or one.*
- *(ii) For each $\mathcal{G}$-measurable random variable Y show that there exists a constant C_Y such that $Y = C_Y$ almost surely.*
- *(iii) Deduce that $\mathbb{P}_{\mathcal{G}}(X) = \mathbb{P}X$ almost surely, for each X in $\mathcal{M}^+(\mathcal{F})$.*
- *(iv) Show for each Borel measurable X that $X(\omega)$ is uniquely determined once we know the values of all $\mathcal{G}$-measurable random variables.*

(8.3) *[This problem gives a version of the Neyman factorization theorem using Kolmogorov conditional expectations. The method of proof is analogous to the method explained in class for the case where conditional distributions exist.] Suppose $\mathbb{P}$ and $\mathbb{P}_\theta$, for $\theta \in \Theta$, are probability measures defined on a sigma-field $\mathcal{F}$, for some index set Θ. Suppose also that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ and that there exist versions of densities*

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} = g_\theta(\omega)h(\omega) \qquad \text{with } g_\theta \in \mathcal{M}^+(\mathcal{G}) \text{ for each } \theta$$

for a fixed $h \in \mathcal{M}^+(\mathcal{F})$ that doesn't depend on θ.

- *(i) Define H to be a version of $\mathbb{P}_{\mathcal{G}}h$. [That is, choose one H from the $\mathbb{P}$-equivalence class of possibilities.] Show that $\mathbb{P}_\theta\{H = 0\} = 0 = \mathbb{P}_\theta\{H = \infty\}$ for each θ.*
- *(ii) For each X in $\mathcal{M}^+(\mathcal{F})$, show that there exists a version of the conditional expectation $\mathbb{P}_\theta(X \mid \mathcal{G})$ that doesn't depend on θ:*

$$\mathbb{P}_\theta(X \mid \mathcal{G}) = \frac{\mathbb{P}_{\mathcal{G}}(Xh)}{H}\{0 < H < \infty\} \qquad \text{a.e. } [\mathbb{P}_\theta] \text{ for every } \theta.$$

*(8.4) *Suppose $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$. Let $X_{\mathcal{G}}$ be a version of $\mathbb{P}_{\mathcal{G}}X$. Define $\mathrm{var}_{\mathcal{G}}(X)$ to equal $\mathbb{P}_{\mathcal{G}}(X - X_{\mathcal{G}})^2$. Show that*

$$\mathrm{var}(X) = \mathbb{P}\big(\mathrm{var}_{\mathcal{G}}X\big) + \mathrm{var}(\mathbb{P}_{\mathcal{G}}X).$$