*(8.1) [conditional Jensen] UGMTTP Problem 5.12.

*(8.2) [See the second Remark on UGMTTP page 127 for the significance of this Problem.] Let $\mathbb{P}$ be Lebsegue measure on $\mathcal{F}$, the Borel sigma-field of $[0, 1]$. Let $\mathcal{G}$ denote the sigma-field generated by all the singletons in $[0, 1]$.

(i) Show that each set in $\mathcal{G}$ is either countable or its complement is countable, and hence it has probability either zero or one.

(ii) For each $\mathcal{G}$-measurable random variable $Y$ show that there exists a constant $C_Y$ such that $Y = C_Y$ almost surely.

(iii) Deduce that $\mathbb{P}_G(X) = \mathbb{P}X$ almost surely, for each $X$ in $\mathcal{M}^+(\mathcal{F})$.

(iv) Show for each Borel measurable $X$ that $X(\omega)$ is uniquely determined once we know the values of all $\mathcal{G}$-measurable random variables.

(8.3) [This problem gives a version of the Neyman factorization theorem using Kolmogorov conditional expectations. The method of proof is analogous to the method explained in class for the case where conditional distributions exist.] Suppose $\mathbb{P}$ and $\mathbb{P}_\theta$, for $\theta \in \Theta$, are probability measures defined on a sigma-field $\mathcal{F}$, for some index set $\Theta$. Suppose also that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ and that there exist versions of densities

$$ \frac{d\mathbb{P}_\theta}{d\mathbb{P}} = g_\theta(\omega) h(\omega) \quad \text{with } g_\theta \in \mathcal{M}^+(\mathcal{G}) \text{ for each } \theta $$

for a fixed $h \in \mathcal{M}^+(\mathcal{F})$ that doesn’t depend on $\theta$.

(i) Define $H$ to be a version of $\mathbb{P}_G h$. [That is, choose one $H$ from the $\mathbb{P}$-equivalence class of possibilities.] Show that $\mathbb{P}_\theta\{H = 0\} = 0 = \mathbb{P}_\theta\{H = \infty\}$ for each $\theta$.

(ii) For each $X$ in $\mathcal{M}^+(\mathcal{F})$, show that there exists a version of the conditional expectation $\mathbb{P}_\theta(X \mid \mathcal{G})$ that doesn’t depend on $\theta$:

$$ \mathbb{P}_\theta(X \mid \mathcal{G}) = \frac{\mathbb{P}_G(XH)}{H} \{0 < H < \infty\} \quad \text{a.e. } [\mathbb{P}_\theta] \text{ for every } \theta. $$

*(8.4) Suppose $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$. Let $X_G$ be a version of $\mathbb{P}_G X$. Define $\text{var}_G(X)$ to equal $\mathbb{P}_G(X - X_G)^2$. Show that

$$ \text{var}(X) = \mathbb{P}(\text{var}_G X) + \text{var}(\mathbb{P}_G X). $$