Statistics 330/600 **Due: Thursday 10 April**
2008: Sheet 9

*(9.1) Suppose $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $Y = \mathbb{P}_{\mathcal{G}} X$ for some sub-sigma-field $\mathcal{G}$ of $\mathcal{F}$. Suppose W is a $\mathcal{G}$-measurable random variable for which $XW \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

(i) Show that $YW \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P}(XW) = \mathbb{P}(YW)$. Hint: Consider $X^{\pm}$ and $W^{\pm}$. Note that Y is defined to equal $\mathbb{P}_{\mathcal{G}} X^+ - \mathbb{P}_{\mathcal{G}} X^-$ almost surely. [By the way: Why is there no $\infty - \infty$ problem in the definition of Y?]

(ii) Show that $Y \geq 0$ almost surely if $\mathbb{P}(XW) \geq 0$ for every bounded, $\mathcal{G}$-measurable, nonnegative W.

*(9.2) Suppose $\{\xi_i : i \in \mathbb{N}\}$ is an exchangeable sequence of random variables, each taking values in $\{0, 1\}$. That is, for each n and each $t = (t_1, \ldots, t_n) \in \{0, 1\}^n$, the probability $\mathbb{P}\{\xi_1 = t_1, \ldots, \xi_n = t_n\}$ depends only on $\sum_{i \leq n} t_i$. Define $S_n = \sum_{i \leq n} \xi_i$. Let $\mathcal{S} = \sigma(S_n)$.

(i) Show that $\mathbb{P}_{\mathcal{S}} \xi_1 = \ldots = \mathbb{P}_{\mathcal{S}} \xi_n$ almost surely.

(ii) Deduce for each $m \leq n$ that $\mathbb{P}_{\mathcal{S}}(S_m/m) = S_n/n$ almost surely.

(9.3) (generalized Polya urn) Suppose an urn initially contains r_0 red balls and $b_0 = N_0 - r_0$ black balls. At step n a ball is selected at random from the urn then thrown back. Let $\xi_n = 1$ if the ball is red, $\xi_n = 0$ otherwise. Another d_n balls of the same color, where $d_n \geq 0$ is a random integer that can depend on the outcomes of the first $n - 1$ draws, are then added to the urn.

Let $R_n = r_0 + \sum_{i \leq n} \xi_i d_i$ denote the number of red balls in the urn, and B_n denote the number of black balls, after completion of the nth step. The total number of balls in the urn is $N_n = R_n + B_n = N_0 + \sum_{i \leq n} d_i$.

(i) Show that R_n/N_n is a martingale with respect to a suitable filtration.

(ii) Is R_n/N_n also a martingale if we allow d_n to take negative values, subject to the constraint $d_n \geq \max(R_{n-1}, B_{n-1})$?

(9.4) [slightly difficult problem] Suppose μ and $\mathbb{P}$ are probability measures on a ***countably generated*** sigma-field $\mathcal{F}_\infty$ on a set Ω. That is, $\mathcal{F}_\infty = \sigma\{F_i : i \in \mathbb{N}\}$. Let π_n denote the partition of Ω generated by the sets $F_1, \ldots, F_n$. (That is, each nonempty member of π_n is of the form $\cap_{i \leq n} A_i$ with $A_i = F_i$ or $A_i = F_i^c$.) Define

$$X_n(\omega) = \sum\nolimits_{B \in \pi_n} \{\omega \in B, \mathbb{P}B > 0\} \frac{\mu B}{\mathbb{P}B}$$
$$Z_n(\omega) = \sum\nolimits_{B \in \pi_n} \{\omega \in B, \lambda B > 0\} \frac{\mu B}{\lambda B} \qquad \text{where } \lambda = \tfrac{1}{2}(\mu + \mathbb{P}).$$

(i) Define $\Omega_n = \cup\{B \in \pi_n : \mathbb{P}B > 0\}$. Show that $\mathbb{P}\Omega_n = 1$ for each n.

(ii) Define $\mathcal{F}_n = \sigma(\pi_n) = \sigma\{F_i : i = 1, \ldots, n\}$. Show that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a $\mathbb{P}$-supermartingale, which converges $\mathbb{P}$ almost surely to some random variable X_∞.

(iii) Show that $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}\}$ is a λ-martingale, which converges λ almost surely to a random variable Z_∞. Show also that Z_∞ is a version of the density $d\mu/d\lambda$.

(iv) Show that there is an $\mathcal{F}_\infty$-measurable subset Ω_0 of $\cap_{n \in \mathbb{N}} \Omega_n$ with $\mathbb{P}\Omega_0 = 1$ such that

$$X_n(\omega) \to X_\infty(\omega) < \infty \qquad \text{and} \qquad \frac{2X_n(\omega)}{1 + X_n(\omega)} = Z_n(\omega) \qquad \text{for each } \omega \in \Omega_0.$$

(v) Deduce that

$$\frac{2X_\infty}{1 + X_\infty} \Omega_0 = Z_\infty \Omega_0 \qquad \text{a.e. } [\lambda]$$

(vi) Deduce that

$$\mathbb{P} \frac{X_\infty \Omega_0 F}{1 + X_\infty} = \mu \frac{\Omega_0 F}{1 + X_\infty} \qquad \text{for each } F \in \mathcal{F}_\infty.$$

(vii) Deduce that

$$\mathbb{P} X_\infty \Omega_0 f = \mu \Omega_0 f \qquad \text{for each } f \in \mathcal{M}^+(\mathcal{F}_\infty).$$

(viii) Conclude that X_∞ is a version of the density $d\mu_0/d\mathbb{P}$, where μ_0 is the part of μ that is absolutely continuous with respect to $\mathbb{P}$.