Solution to HW 1.4

The set $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is called the ***extended real line****. Write $\mathcal{A}$ for the sigma-field on $\overline{\mathbb{R}}$ generated by $\mathcal{B}(\mathbb{R})$ together with the two singleton sets $\{-\infty\}$ and $\{\infty\}$. Show that $\mathcal{A}$ is also generated by $\mathcal{E} = \{\,[-\infty, t] : t \in \mathbb{R}\}$.*

Let me abbreviate $\mathcal{B}(\mathcal{R})$ to $\mathcal{B}$ and write $\mathcal{F}$ for $\{\{-\infty\}, \{+\infty\}\} \cup \mathcal{B}$. There are two notational subtleties to be careful about. For a subset A of $\mathbb{R}$, should A^c refer to $\mathbb{R}\backslash A$ or $\overline{\mathbb{R}}\backslash A$? It is better to avoid the A^c notation for this problem. Also, if $\mathcal{D}$ is a collection of subsets of $\mathbb{R}$, should $\sigma(\mathcal{D})$ refer to a sigma-field on $\mathbb{R}$ or a sigma-field on $\overline{\mathbb{R}}$? To avoid ambiguity, let $\overline{\sigma}(\mathcal{D})$ denote the smallest sigma-field $\overline{\mathcal{A}}$ on $\overline{\mathbb{R}}$ for which $\mathcal{D} \subseteq \overline{\mathcal{A}}$ and $\sigma(\mathcal{D})$ denote the smallest sigma-field $\mathcal{A}$ on $\mathbb{R}$ for which $\mathcal{D} \subseteq \mathcal{A}$. For example, $\sigma(\mathcal{B}) = \mathcal{B} \neq \overline{\sigma}(\mathcal{B})$.

In order to prove that $\overline{\sigma}(\mathcal{E}) = \overline{\sigma}(\mathcal{F})$ it suffices to prove $\mathcal{F} \subseteq \overline{\sigma}(\mathcal{E})$ and $\mathcal{E} \subseteq \overline{\sigma}(\mathcal{F})$.

One inclusion is easy: for each real t,

$$[-\infty, t] = \{-\infty\} \cup (-\infty, t],$$

a union of two sets in the sigma-field $\overline{\sigma}(\mathcal{F})$. Thus $\mathcal{E} \subseteq \overline{\sigma}(\mathcal{F})$.

For the other inclusion, first note that

$$\{-\infty\} = \cap_{n\in\mathbb{N}}[-\infty, -n] \in \overline{\sigma}(\mathcal{E})$$
$$\{+\infty\} = \big(\cup_{n\in\mathbb{N}}\,[-\infty, n]\big)^c \in \overline{\sigma}(\mathcal{E})$$

To show that $\mathcal{B} \subseteq \overline{\sigma}(\mathcal{E})$ argue indirectly. Define

$$\mathcal{B}_0 = \{B \in \mathcal{B} : B \in \overline{\sigma}(\mathcal{E})\} = \mathcal{B} \cap \overline{\sigma}(\mathcal{E}).$$

If $B \in \mathcal{B}_0$ then the fact that $\mathcal{B}$ is a sigma-field on $\mathbb{R}$ implies that $\mathbb{R}\backslash B \in \mathcal{B}$. Also $\mathbb{R}\backslash B = (\overline{\mathbb{R}}\backslash B) \cap \{\{-\infty\}, \{+\infty\}\}^c \in \overline{\sigma}(\mathcal{E})$. Thus $\mathbb{R}\backslash B \in \mathcal{B}_0$.

If $\{B_i : i \in \mathbb{N}\} \subseteq \mathcal{B}_0$ then $\cup_i B_i$ is a member of $\mathcal{B}$ (because $\mathcal{B}$ is a sigma-field on $\mathbb{R}$) and a member of the sigma-field $\overline{\sigma}(\mathcal{E})$ on $\overline{\mathbb{R}}$. That is, $\cup_i B_i \in \mathcal{B}_0$. Thus $\mathcal{B}_0$ is a sigma-field on $\mathbb{R}$. (Should I also check that $\emptyset \in \mathcal{B}_0$?)

Finally, note that

$$\mathcal{B}_0 \supseteq \mathcal{E}_0 = \{(-\infty, t] : t \in \mathbb{R}\}$$

because $(-\infty, t] \in \mathcal{B}$ and $(-\infty, t] = [-\infty, t] \cap \{-\infty\}^c \in \overline{\sigma}(\mathcal{E})$. It follows that

$$\mathcal{B} = \sigma(\mathcal{E}_0) \subseteq \sigma(\mathcal{B}_0) = \mathcal{B}_0$$

so that $\mathcal{B} \subseteq \overline{\sigma}(\mathcal{E})$.