Statistics 330b/600b, spring 2009
Homework # 1
Due: Thursday 22 January

Please attempt at least the starred problems.

*[1] Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For $B \subseteq \mathcal{Y}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. For $A \subseteq \mathcal{X}$ define $T(A) := \{T(x) : x \in A\}$. Some of the following assertions are true in general and some are false.

$$T\big(\cup_i A_i\big) = \cup_i T(A_i) \qquad \text{and} \qquad T^{-1}\big(\cup_i B_i\big) = \cup_i T^{-1}(B_i)$$
$$T\big(\cap_i A_i\big) = \cap_i T(A_i) \qquad \text{and} \qquad T^{-1}\big(\cap_i B_i\big) = \cap_i T^{-1}(B_i)$$
$$T\big(A^c\big) = \big(T\left(A\right)\big)^c \qquad \text{and} \qquad T^{-1}\big(B^c\big) = \big(T^{-1}\left(B\right)\big)^c$$
$$T^{-1}\big(T(A)\big) = A \qquad \text{and} \qquad T\big(T^{-1}(B)\big) = B$$

Provide counterexamples for each of the false assertions.

*[2] Let $\Omega = \{0,1\}^{\mathbb{N}}$. For each n in $\mathbb{N}$ let $\mathcal{E}_n$ denote the collection subsets of Ω of the form $A(a) = \{\omega \in \Omega : \omega_i = a_i \text{ for } i = 1, \dots, n\}$, where $a = (a_1, \dots, a_n)$ ranges over the 2^n elements of $\{0,1\}^n$. Let $\mathcal{E} = \cup_{n\in\mathbb{N}}\mathcal{E}_n$. Write $\mathcal{F}$ for the generated sigma-field, $\mathcal{F} = \sigma(\mathcal{E})$. And let $\mathbf{m}$ denote Lebesgue measure on $\mathcal{B}(0,1]$.

For each n in $\mathbb{N}$ define a function $X_n : (0,1] \to \{0,1\}$ by putting $X_n(u) = 1$ when u lies in the union of the 2^{n-1} intervals $(\,(k-1)/2^n, k/2^n]$, for $k = 2,4,6,\dots,2^n$, and $X_n(u) = 0$ otherwise. Define a function $T : (0,1] \to \Omega$ by letting $T(u)$ have nth coordinate $X_n(u)$.

(i) Show that T is $\mathcal{B}(0,]\backslash\mathcal{F}$-measurable. Hint: No need to reprove the result from Example 2.7, which I discussed in class.

(ii) Let $\mathbb{P} = T(\mathbf{m})$, the image of $\mathbf{m}$ under T. Show that $\mathbb{P}$ is a probability measure on $\mathcal{F}$ for which $\mathbb{P}E = 2^{-n}$ for each E in $\mathcal{E}_n$.

[3] Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, which is equipped with a σ-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which T is $\sigma(T)\backslash\mathcal{B}$-measurable. Show that to each f in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a $\mathcal{B}\backslash\mathcal{B}[0,\infty]$-measurable function g from $\mathcal{Y}$ into $[0,\infty]$ such that $f(x) = g(T(x))$, for all x in $\mathcal{X}$, by following these steps.

(i) Consider the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(ii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to f. Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property.

(iii) In part (ii), why can't we assume that $\lim g_n(y)$ exists for each y?

[4] Suppose a set $\mathcal{E}$ of subsets of $\mathcal{X}$ cannot separate a particular pair of points x, y, that is, for every E in $\mathcal{E}$, either $\{x,y\} \subseteq E$ or $\{x,y\} \subseteq E^c$. Show that $\sigma(\mathcal{E})$ also cannot separate the pair.