Suppose $T$ maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For $B \subseteq \mathcal{Y}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. For $A \subseteq \mathcal{X}$ define $T(A) := \{T(x) : x \in A\}$. Some of the following assertions are true in general and some are false.

\[
T \left( \bigcup_i A_i \right) = \bigcup_i T(A_i) \quad \text{and} \quad T^{-1} \left( \bigcup_i B_i \right) = \bigcup_i T^{-1}(B_i)
\]

\[
T \left( \bigcap_i A_i \right) = \bigcap_i T(A_i) \quad \text{and} \quad T^{-1} \left( \bigcap_i B_i \right) = \bigcap_i T^{-1}(B_i)
\]

\[
T \left( A^c \right) = (T(A))^c \quad \text{and} \quad T^{-1} \left( B^c \right) = (T^{-1}(B))^c
\]

\[
T^{-1} (T(A)) = A \quad \text{and} \quad T \left( T^{-1}(B) \right) = B
\]

Provide counterexamples for each of the false assertions.

Let $\Omega = \{0, 1\}^\mathbb{N}$. For each $n$ in $\mathbb{N}$ let $E_n$ denote the collection subsets of $\Omega$ of the form $A(a) = \{\omega \in \Omega : \omega_i = a_i \text{ for } i = 1, \ldots, n\}$, where $a = (a_1, \ldots, a_n)$ ranges over the $2^n$ elements of $\{0, 1\}^n$. Let $\mathcal{E} = \cup_{n \in \mathbb{N}} E_n$. Write $\mathcal{F}$ for the generated sigma-field, $\mathcal{F} = \sigma(\mathcal{E})$. And let $m$ denote Lebesgue measure on $\mathcal{B}(0, 1]$.

For each $n$ in $\mathbb{N}$ define a function $X_n : (0, 1] \to \{0, 1\}$ by putting $X_n(u) = 1$ when $u$ lies in the union of the $2^{n-1}$ intervals $\left((k-1)/2^n, k/2^n\right]$, for $k = 2, 4, 6, \ldots, 2^n$, and $X_n(u) = 0$ otherwise. Define a function $T : (0, 1] \to \Omega$ by letting $T(u)$ have $n$th coordinate $X_n(u)$.

(i) Show that $T$ is $\mathcal{B}(0, 1] \setminus \mathcal{F}$-measurable. Hint: No need to reprove the result from Example 2.7, which I discussed in class.

(ii) Let $\mathbb{P} = T(m)$, the image of $m$ under $T$. Show that $\mathbb{P}$ is a probability measure on $\mathcal{F}$ for which $\mathbb{P}E = 2^{-n}$ for each $E$ in $\mathcal{E}_n$.

Suppose $T$ is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, which is equipped with a $\sigma$-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which $T$ is $\sigma(T) \setminus \mathcal{B}$-measurable. Show that to each $f$ in $\mathcal{M}_{\text{simple}}^+(\mathcal{X}, \sigma(T))$ there exists a $\mathcal{B} \setminus \mathcal{B}[0, \infty]$-measurable function $g$ from $\mathcal{Y}$ into $[0, \infty]$ such that $f(x) = g(T(x))$, for all $x$ in $\mathcal{X}$, by following these steps.

(i) Consider the case where $f \in \mathcal{M}_{\text{simple}}^+(\mathcal{X}, \sigma(T))$.

(ii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}_{\text{simple}}^+(\mathcal{X}, \sigma(T))$ that increases pointwise to $f$. Define $g(y) = \lim \sup g_n(y)$ for each $y$ in $\mathcal{Y}$. Show that $g$ has the desired property.

(iii) In part (ii), why can’t we assume that $\lim g_n(y)$ exists for each $y$?

Suppose a set $\mathcal{E}$ of subsets of $\mathcal{X}$ cannot separate a particular pair of points $x, y$, that is, for every $E$ in $\mathcal{E}$, either $\{x, y\} \subseteq E$ or $\{x, y\} \subseteq E^c$. Show that $\sigma(\mathcal{E})$ also cannot separate the pair.