Let $\mathcal{E}_0$ be a countable collection of subsets of some set $X$. For each $n$ in $\mathbb{N}$, define $\mathcal{E}_n$ to consist of all sets of the form: a complement of a set from $\mathcal{E}_{n-1}$; or a union of finitely many sets from $\mathcal{E}_{n-1}$; or an intersection of finitely many sets from $\mathcal{E}_{n-1}$. Define $\mathcal{E}^* := \bigcup_{n \in \mathbb{N}} \mathcal{E}_n$. Show that $\mathcal{E}^*$ is a countable field of subsets of $X$ (cf. HW3.1). Also show that $\sigma(\mathcal{E}_0) = \sigma(\mathcal{E}_0)$.

* [2] Suppose $A$ is a countably generated sigma-field on a set $X$. That is, $A = \sigma(\mathcal{E})$ for some countable collection of subsets. By Problem 1, you may assume that $\mathcal{E}$ is a field. Suppose also that every singleton subset in $\mathcal{A}$, that is, $\{x\} \in A$ for each $x \in X$. Show that $\Delta := \{(x, y) \in X \times X : x = y\}$ belongs to the product sigma-field $A \otimes A$. Hint: Use HW1.4 to show, for each pair $x \neq y$ in $X$, that there exists a set $E \in \mathcal{E}$ with $x \in E$ and $y \in E^c$. Consider the collection of subsets $\{E \times E^c : E \in \mathcal{E}\}$.

* [3] Let $P$ and $Q$ be probability measures on $(X, \mathcal{A})$ with densities $p(x)$ and $q(x)$ with respect to some measure $\mu$. Define $\alpha(x) = \min (1, q(x)/p(x))$ and $\alpha_0 = \mu(p \land q)$ and let $\lambda$ denote the measure with density $(q - p)^+$ with respect to $\mu$. For each $x \in X$ define

$$\gamma_x = \alpha(x) \delta_x + \frac{1 - \alpha(x)}{1 - \alpha_0} \lambda$$

where $\delta_x$ denotes the point mass at $x$. That is,

$$\gamma^y_x h(y) = \alpha(x) h(x) + \frac{1 - \alpha(x)}{1 - \alpha_0} \lambda h \quad \text{for each } h \in M^+(X, \mathcal{A})$$

Let $\Gamma = \{\gamma_x : x \in X\}$ and $P = P \otimes \Gamma$.

(i) I have ignored the possibility $\alpha_0 = 1$. Show $P = Q$ if $\alpha_0 = 1$. [You might like to modify the calculations that follow to cover this trivial case.]

(ii) Show that $P \alpha(x) = \alpha_0$ and $\lambda X = 1 - \alpha_0 = \|P - Q\|_{TV}$.

(iii) Show that each $\gamma_x$ is a probability measure on $\mathcal{A}$ and the map $x \rightarrow \lambda^y_x h(y)$ is $\mathcal{A}$-measurable for each $h \in M^+(X, \mathcal{A})$.

(iv) Show that $\mathbb{P}^{x,y} h(x) = Ph$ and $\mathbb{P}^{x,y} h(y) = Qh$ for each $h \in M^+(X, \mathcal{A})$. That is, $\mathbb{P}$ has marginals $P$ and $Q$.

(v) Let $\Delta$ denote the (indicator function of) the diagonal, $\{(x, y) \in X \times X : x = y\}$. Assume $\Delta \in A \otimes A$. Show that $\mathbb{P}\{x \neq y\} = \|P - Q\|_{TV}$. Hint: Prove $P^x \lambda^y (1 - \alpha(x)) \Delta(x, y) = 0$. You may invoke Tonelli’s Theorem for $\mu \otimes \mu$ if you feel the need to.

[4] Let $X$ be a topological space for which the collection $\mathcal{G}$ of all open subsets has a countable base. That is, there exists a countable subcollection $\mathcal{G}_0$ of $\mathcal{G}$ such that every set in $\mathcal{G}$ can be written as a union of subsets in $\mathcal{G}_0$. Equip $X \times X$ with its product topology: the open subsets of $X \times X$ are defined to be those subsets expressible as a union of sets of the form $G \times H$, with $G, H \in \mathcal{G}$. Show that $\mathcal{B}(X \times X) = \mathcal{B}(X) \otimes \mathcal{B}(X)$. Remark: You could also prove the more general result for the product of two different topological spaces with each topology having a countable base.