Let $\lambda$ be a probability measure on the product sigma-field $\mathcal{A} \otimes \mathcal{B}$ of a product space $X \times Y$, with $X$-marginal $P$ and $Y$-marginal $Q$. Suppose $\mathcal{P} = \{P_y : y \in Y\}$ is a Markov kernel for which $\lambda = Q \otimes \mathcal{P}$.

(i) Suppose $A = \sigma(E)$ for some countable $E \subseteq A$. Without loss of generality suppose $E$ is stable under the formation of pairwise intersections. Suppose $\overline{\mathcal{P}} = \{P_y : y \in Y\}$ is another Markov kernel for which $\lambda = Q \otimes \overline{\mathcal{P}}$. Show that $P_y = \overline{P}_y$, as measures on $A$, a.e.$[Q]$. Hint: Consider $\lambda(E \times B)$ for various $E \in \mathcal{E}$ and $B \in \mathcal{B}$.

(ii) If $\Delta$ is an $\mathcal{A} \otimes \mathcal{B}$-measurable set with $\lambda \Delta = 0$, show that $P_y\{x \in X : (x, y) \notin \Delta\} = 0$ a.e.$[Q]$.

(iii) Suppose $Y$ is an $\mathcal{A}\setminus\mathcal{B}$-measurable map from $X$ into $Y$. Define $\Delta = \{(x, y) \in X \times Y : Y(x) = y\}$. Suppose $\Delta \in \mathcal{A} \otimes \mathcal{B}$ and $\lambda \Delta = 1$. Show that $P_y\{x \in X : Y(x) \neq y\} = 0$ a.e.$[Q]$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $i = 1, 2$, suppose $X_i$ is an $\mathcal{A}\setminus\mathcal{B}_i$-measurable map from $\Omega$ into $X_i$, where $\mathcal{B}_i$ is a sigma-field on $X_i$. Equip $X = X_1 \times X_2$ with its product sigma-field $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$. Suppose $X_1$ has distribution $P_1$.

From HW6.1, you know that $T(\omega) = (X_1(\omega), X_2(\omega))$ is $\mathcal{F}\setminus\mathcal{B}$-measurable. If $X_1$ and $X_2$ are independent, show that $T$ has distribution $P_1 \otimes P_2$.

Let $m$ denote Lebesgue measure on $\mathcal{B}(\mathbb{R}^2)$. For a fixed $t > 0$ define another measure $\lambda$ on $\mathcal{B}(\mathbb{R}^2)$ by $\lambda(D) = \int_D m[D \times [c, d]]$ for $D \subset [a, b] \times [c, d]$.

(i) Show that $mD = \lambda D$ for each set $D$ of the form $[a, b] \times [c, d]$.

(ii) Use a generating class argument to deduce that $m = \lambda$, as measures on $\mathcal{B}(\mathbb{R}^2)$. [Note that $m(\mathbb{R}^2) = \infty$. If you plan on using some sort of $\pi$--$\lambda$ argument you will need to work on chunks of $\mathbb{R}^2$ with finite measure.]

Let $\mathbb{P}$ be the uniform distribution on the unit ball $B = \{z \in \mathbb{R}^2 : |z| \leq 1\}$. That is, $\mathbb{P}f = \pi^{-1}m^2 f(z)\{\{z\}| \leq 1\}$ for each $f \in M^+(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, where $m$ denotes Lebesgue measure on $\mathcal{B}(\mathbb{R}^2)$. Define $R(z) = |z|$ and $\psi(z) = z/|z|$ with $\psi(0) = 0$.

(i) Prove that $R$ and $\psi$ are independent. Hint: Consider $R(z)\{\{z\}| \leq t\}$ for $0 < t < 1$. Use Problem [3] to factorize into $t^2\mathbb{P}g(z/|z|)$. Then what?

(ii) Write $Q$ for the distribution of $R(z)$ under $\mathbb{P}$. Let $\mathbb{P}_t$ denote the uniform distribution on $\{z \in \mathbb{R}^2 : |z| = t\}$. [You may regard $\mathbb{P}_t$ as the distribution of $tv(z)$.] Show that $\mathbb{P}f(z) = Q^t \mathbb{P}_t f(z)$ for a suitably large class of functions $f$. [That is, the conditional distribution $\mathbb{P}(\cdot \mid R = t)$ equals $\mathbb{P}_t$ a.e.$[Q]$.]