Please attempt at least the starred problems.

*[1] Suppose \(\mathcal{G}\) is a sub-sigma-field of \(\mathcal{F}\) for which either \(\mathbb{P}\mathcal{G} = 0\) or \(\mathbb{P}\mathcal{G} = 1\) for every \(G \in \mathcal{G}\).

(i) For each \(Z \in \mathcal{M}^+(\mathcal{G})\), show that \(\mathbb{P}\{Z = c\} = 1\) for some constant \(c \in [0, \infty]\).

(ii) For each \(X \in \mathcal{M}^+(\mathcal{F})\), show that \(\mathbb{P}_{\mathcal{G}}(X) = \mathbb{P}X \ \text{a.e.} \ \mathbb{P}\).

*[2] Suppose \(\{\xi_i : i \in \mathbb{N}\}\) are independent random variables with \(\mathbb{P}\{\xi_i = 1\} = 1/2 = \mathbb{P}\{\xi_i = -1\}\), for each \(i\). Suppose \(X_0 \equiv 1\) and \(X_n = X_0 + \xi_1 + \cdots + \xi_n\). Let \(\mathcal{F}_n = \sigma\{X_0, \xi_1, \ldots, \xi_n\}\). Define \(\tau = \inf\{n : X_n = 0\}\) and \(Z_n = X_{\tau \wedge n}\).

(i) Show that \((X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\) is a martingale. Does \(X_n\) converge almost surely?

(ii) Show that \((Z_n, \mathcal{F}_n) : n \in \mathbb{N}_0\) is a nonnegative martingale.

(iii) Show that \(Z_{n+1} - Z_n \to 0\) almost surely. Hint: What do you know about sequences of real numbers that converge to finite limits?

(iv) From part (iii) and the fact that \(|X_{n+1} - X_n| = 1\), deduce that \(\tau < \infty\) almost surely.

(v) Show that \(\mathbb{P}X_{\tau}\{\tau < \infty\} \neq \mathbb{P}X_0\). Why does this fact not contradict the Stopping Time Lemma?

*[3] Suppose \((X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\) is a martingale for which \(\sup_{n \in \mathbb{N}_0} \mathbb{P}X_n^2 < \infty\). Write \(X_n\) as a sum of increments, \(X_n = X_0 + \sum_{1 \leq i \leq n} \xi_i\).

(i) Show that \(\mathbb{P}X_n^2 = \mathbb{P}X_0^2 + \sum_{1 \leq i \leq n} \mathbb{P}\xi_i^2\).

(ii) Show that \(\{X_n\}\) is a Cauchy sequence in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) and hence \(\mathbb{P}|X_n - Z|^2 \to 0\) as \(n \to \infty\) for some \(Z\) in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\).

(iii) For each \(\epsilon > 0\), show that
\[
\mathbb{P}\{\sup_{k \geq 1} |Z_{n+k} - Z_n| > \epsilon\} \leq \epsilon^{-2} \sum_{k \geq n+1} \mathbb{P}\xi_k^2 \to 0 \quad \text{as} \ n \to \infty.
\]

Hint: Start with a maximum over \(1 \leq k \leq m\) then let \(m\) tend to infinity. You may use without proof any result established in class.

(iv) Define \(A_n(\epsilon) = \{\sup_{\ell, m \geq n} |X_{\ell} - X_m| > 2\epsilon\}\). Show that the set \(A = \bigcup_{k \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} A_n(1/k)\) has zero probability.

(v) Show that \(X_n(\omega)\) is a Cauchy sequence for each \(\omega\) in \(A^c\).

(vi) Show that \(X_n \to Z\) almost surely.

[4] (hard) Suppose \(\{Z_n : n \in \mathbb{N}_0\}\) is a sequence of random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Define a filtration by \(\mathcal{F}_n = \sigma(Z_i : i \leq n)\). Suppose \(\tau\) is a stopping time with respect to this filtration. Define \(X_i = Z_{i\wedge \tau}\). Show that \(\mathcal{F}_\tau = \sigma(X_i : i \in \mathbb{N}_0)\). Hint: It helped me to define \(\mathcal{G}_n = \sigma(X_i : i \leq n)\) and \(\mathcal{G}_\infty = \sigma(\cup_{n \in \mathbb{N}_0} \mathcal{G}_i)\). It is not so hard to show \(\mathcal{G}_\infty \subseteq \mathcal{F}_\tau\). For the other inclusion, I found it helpful to start by showing that \(\tau\) is also a stopping time for the the \(\{\mathcal{G}_n : n \in \mathbb{N}_0\}\) filtration.