Statistics 330b/600b, spring 2009
Homework # 9
Due: Thursday 9 April

Please attempt at least the starred problems.

*[1] UGMTP Problem 6.6. Notice that the stopping times might take the value $+\infty$. You will need suitable limiting arguments after invoking the Stopping Time Lemma with $\sigma \wedge N$ and $\tau \wedge N$, for a finite N.

*[2] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\mathcal{F}$ countably generated. In class you saw that there is a filtration $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$ with each $\mathcal{F}_n$ generated by a finite set $\mathcal{A}_n$ of atoms and such that $\mathcal{F} = \sigma\left(\cup_{n\in\mathbb{N}_0}\mathcal{F}_n\right)$.

Suppose μ is another probability measure on $\mathcal{F}$, not necessarily dominated by $\mathbb{P}$. Define $\mathbb{Q} = \frac{1}{2}(\mathbb{P} + \mu)$ and

$$X_n(\omega) = \sum\nolimits_{A\in\mathcal{A}_n} \{\mathbb{Q}A > 0\}\frac{\mu A}{\mathbb{Q}A}\{\omega \in A\}$$
$$Z_n(\omega) = \sum\nolimits_{A\in\mathcal{A}_n} \{\mathbb{P}A > 0\}\frac{\mu A}{\mathbb{P}A}\{\omega \in A\}.$$

From the arguments presented in class, you know that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a $\mathbb{Q}$-martingale. Also there exists a random variable X_∞ with $0 \le X_\infty(\omega) \le 2$ for every ω such $X_n \to X_\infty$ a.e. $[\mathbb{Q}]$ and $\mathbb{Q}|X_n - X_\infty| \to 0$. Moreover, X_∞ is (a version of) the density $d\mu/d\mathbb{Q}$.

(i) Show that $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a nonnegative $\mathbb{P}$-supermartingale. Deduce that there exists a random variable Z_∞ with $0 \le Z_\infty(\omega) < \infty$ for every ω such that $Z_n \to Z_\infty$ a.e. $[\mathbb{P}]$.

(ii) Define $\Omega_n = \cup\{A \in \mathcal{A}_n : \mathbb{P}A > 0\}$ and $\Omega' = \cap_{n\in\mathbb{N}_0}\Omega_n$. Show that $\mathbb{P}\Omega' = 1$ and

$$X_n(\omega) = \frac{2Z_n(\omega)}{1 + Z_n(\omega)} \qquad \text{for all } \omega \text{ in } \Omega'.$$

(iii) Deduce that there is a subset Ω'' of Ω' with $\mathbb{P}\Omega'' = 1$ for which

$$X_\infty(\omega) = \frac{2Z_\infty(\omega)}{1 + Z_\infty(\omega)} \qquad \text{for all } \omega \text{ in } \Omega''.$$

(iv) For each f in $\mathcal{M}^+(\omega, \mathcal{F})$ show that

$$\mu\left(\Omega'' f\right) = \mathbb{P}\left(\Omega'' f Z_\infty\right).$$

Hint: If you are planning on any subtractions it might be safer to start with f bounded.

(v) Conclude that Z_∞ is (a version of) the density $d\mu_0/d\mathbb{P}$, where μ_0 is the part of μ that is dominated by $\mathbb{P}$. That is, if μ_0 is the restriction of μ to Ω'' and μ_1 is the restriction of μ to $(\Omega'')^c$ then μ_1 and $\mathbb{P}$ are mutually singular and $\mu_0 \ll \mathbb{P}$ with density Z_∞. You might even want to take $Z_\infty(\omega) = 0$ if $\omega \in (\Omega'')^c$. [Compare with the Lebesgue decomposition of μ.]