Please attempt at least the starred problems.

*1* UGMTP Problem 6.6. Notice that the stopping times might take the value $+\infty$. You will need suitable limiting arguments after invoking the Stopping Time Lemma with $\sigma \wedge N$ and $\tau \wedge N$, for a finite $N$.

*2* Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\mathcal{F}$ countably generated. In class you saw that there is a filtration $\{\mathcal{F}_n : n \in \mathbb{N}_0\}$ with each $\mathcal{F}_n$ generated by a finite set $A_n$ of atoms and such that $\mathcal{F} = \sigma(\bigcup_{n \in \mathbb{N}_0} \mathcal{F}_n)$.

Suppose $\mu$ is another probability measure on $\mathcal{F}$, not necessarily dominated by $P$. Define $Q = \frac{1}{2}(P + \mu)$ and

$$X_n(\omega) = \sum_{A \in A_n} \{QA > 0\} \frac{\mu A}{QA} \{\omega \in A\}$$

$$Z_n(\omega) = \sum_{A \in A_n} \{PA > 0\} \frac{\mu A}{PA} \{\omega \in A\}.$$

From the arguments presented in class, you know that $\{(X_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a $Q$-martingale. Also there exists a random variable $X_\infty$ with $0 \leq X_\infty(\omega) \leq 2$ for every $\omega$ such $X_n \rightarrow X_\infty$ a.e. $[Q]$ and $Q|X_n - X_\infty| \rightarrow 0$. Moreover, $X_\infty$ is (a version of) the density $d\mu/dQ$.

(i) Show that $\{(Z_n, \mathcal{F}_n) : n \in \mathbb{N}_0\}$ is a nonnegative $P$-supermartingale. Deduce that there exists a random variable $Z_\infty$ with $0 \leq Z_\infty(\omega) < \infty$ for every $\omega$ such that $Z_n \rightarrow Z_\infty$ a.e. $[P]$.

(ii) Define $\Omega_n = \bigcup\{A \in A_n : PA > 0\}$ and $\Omega' = \cap_{n \in \mathbb{N}_0} \Omega_n$. Show that $P\Omega' = 1$ and

$$X_n(\omega) = \frac{2Z_n(\omega)}{1 + Z_n(\omega)} \quad \text{for all } \omega \text{ in } \Omega'.$$

(iii) Deduce that there is a subset $\Omega''$ of $\Omega'$ with $P\Omega'' = 1$ for which

$$X_\infty(\omega) = \frac{2Z_\infty(\omega)}{1 + Z_\infty(\omega)} \quad \text{for all } \omega \text{ in } \Omega''.$$

(iv) For each $f$ in $M^+(\omega, \mathcal{F})$ show that

$$\mu(\Omega'' f) = P(\Omega'' f Z_\infty).$$

Hint: If you are planning on any subtractions it might be safer to start with $f$ bounded.

(v) Conclude that $Z_\infty$ is (a version of) the density $d\mu_0/dP$, where $\mu_0$ is the part of $\mu$ that is dominated by $P$. That is, if $\mu_0$ is the restriction of $\mu$ to $\Omega''$ and $\mu_1$ is the restriction of $\mu$ to $(\Omega'')^c$ then $\mu_1$ and $P$ are mutually singular and $\mu_0 \ll P$ with density $Z_\infty$. You might even want to take $Z_\infty(\omega) = 0$ if $\omega \in (\Omega'')^c$. [Compare with the Lebesgue decomposition of $\mu$.]