Statistics 330b/600b, spring 2010
Homework # 1
Due: Thursday 21 January

Please attempt at least the starred problems.

*[1] Suppose $\mathcal{A}$ is a sigma-field of subsets of $\mathcal{X}$ and $\mathcal{N}$ is a set of subsets of $\mathcal{X}$ that is stable under countable unions. Define $\mathcal{B} := \{B \subseteq \mathcal{X} : B \Delta A \subseteq N \text{ for some } A \in \mathcal{A} \text{ and } N \in \mathcal{N}\}$. Show that $\mathcal{B}$ is a sigma-field.

*[2] Let $\Omega = \{0,1\}^{\mathbb{N}}$. For each n in $\mathbb{N}$ let $\mathcal{C}_n$ denote the set of subsets of Ω that are cylinder sets with base in $\{0,1\}^n$, that is, sets of the form

$$C = \{\omega \in \Omega : (\omega_1, \dots, \omega_n) \in A\} \qquad \text{for some } A \in \{0,1\}^n.$$

(Note that $\#\mathcal{C}_n = 2^n$.) Define $\mathcal{E} = \cup_{n \in \mathbb{N}} \mathcal{C}_n$.

(i) Show that $\mathcal{E}$ is a field of subsets of Ω. (That is, $\emptyset \in \mathcal{E}$ and $\mathcal{E}$ is stable under complements and finite unions.)

(ii) If $E = \{\omega \in \Omega : (\omega_1, \dots, \omega_n) \in A\} \in \mathcal{E}$ define $\mathbb{P}E = \#A/2^n$. Show that $\mathbb{P}E$ is the same for each possible representation of E as a cylinder set. Note: The ambiguity is: $E = \{\omega \in \Omega : (\omega_1, \dots, \omega_m) \in A \times \{0,1\}^{m-n}\}$ for each $m > n$.

(iii) Show that $\mathbb{P}$ is a finitely additive measure on $\mathcal{E}$ with $\mathbb{P}\Omega = 1$.

(iv) Show that $\mathbb{P}E_n \downarrow 0$ for each decreasing sequence of sets $\{E_n\}$ in $\mathcal{E}$ with $\cap_n E_n = \emptyset$.

(v) Define $X_i(\omega) = \omega_i$ for $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. Show that each X_i is $\sigma(\mathcal{C})\backslash\mathcal{B}(\mathbb{R})$-measurable.

(vi) Show that

$$\mathbb{P}\{X_1 \in A_1, \dots, X_n \in A_n\} = \prod\nolimits_{i \le n} \mathbb{P}\{X_i \in A_i\} = 2^{-n} \qquad \text{for each choice of } A_i \subseteq \{0,1\}.$$

(vii) Define $Y(\omega) = \sum_{i \in \mathbb{N}} 2^{-i} X_i(\omega)$. Show that $\mathbb{P}\{\omega : Y(\omega) \le y\} = y$ for each y in $[0,1]$. Hint: Start with y a dyadic rational.

*[3] Suppose T maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For $B \subseteq \mathcal{Y}$ define

$$T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}.$$

For $A \subseteq \mathcal{X}$ define $T(A) := \{T(x) : x \in A\}$. Some of the following assertions are true in general and some are false.

$$\begin{aligned} T\big(\cup_i A_i\big) &= \cup_i T(A_i) &\quad \text{and} \quad & T^{-1}\big(\cup_i B_i\big) = \cup_i T^{-1}(B_i) \\ T\big(\cap_i A_i\big) &= \cap_i T(A_i) &\quad \text{and} \quad & T^{-1}\big(\cap_i B_i\big) = \cap_i T^{-1}(B_i) \\ T\big(A^c\big) &= \big(T\left(A\right)\big)^c &\quad \text{and} \quad & T^{-1}\big(B^c\big) = \big(T^{-1}\left(B\right)\big)^c \\ T^{-1}\big(T(A)\big) &= A &\quad \text{and} \quad & T\big(T^{-1}(B)\big) = B \end{aligned}$$

Provide counterexamples for each of the false assertions.

[4] In class I asserted that $(\cup_{i \in I} A_i) \Delta (\cup_{i \in I} B_i) = \max_{i \in I} |A_i - B_i|$ but I only showed $\le$. Give an example where equality fails.