Statistics 330b/600b, spring 2010
Homework # 10
Due: Thursday 15 April

For this sheet, suppose $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $(\mathcal{T}, \mathcal{B}, Q)$ *are two probability spaces and* T *is an* $\mathcal{F}\backslash\mathcal{B}$*-measurable map from* Ω *into* $\mathcal{B}$ *that has distribution* Q*.*

*[1] Let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-sigma-field of $\mathcal{F}$ containing all $\mathbb{P}$-negligible sets. Suppose $\mathbb{P}(XW) = 0$ for every bounded random variable W for which $\mathbb{P}_{\mathcal{G}}W = 0$ almost surely. Show that X must be $\mathcal{G}$-measurable, by the following steps.

(i) If $F \in F$ and $F = Z$ almost surely for some $\mathcal{G}$-measurable random variable Z, why must F actually belong to $\mathcal{G}$?

(ii) Show that the set $T = \{t \in \mathbb{R} : \mathbb{P}\{X = t\} > 0\}$ is at worst countable.

(iii) For each fixed t in $\mathbb{R}\backslash T$ define $Z_t = \mathbb{P}_{\mathcal{G}}\{X > t\}$. Why must we have $0 \leq Z_t \leq 1$ almost surely?

(iv) For t as in the previous part, define $W_t = \{X > t\} - Z_t$. Explain why $\mathbb{P}(X - t)W_t = 0$ and why $(X - t)W_t \geq 0$ almost surely. Deduce that $\{X > t\} \in \mathcal{G}$.

(v) Explain why X must be $\mathcal{G}$-measurable.

[2] (convergence of normals) UGMTP Problem 9.11.

*[3] Suppose X_n has a $N(\mu_n, \sigma_n^2)$ distribution, and $X_n \rightsquigarrow P$, for some probability measure P on $\mathcal{B}(\mathbb{R})$.

(i) Explain why there exists a finite constant M such that $\liminf \mathbb{P}\{|X_n| > M\} > 3/4$.

(ii) Explain why we must have $|\mu_n| \leq M$ and $\sigma_n \leq 8M/(3\sqrt{2\pi})$ eventually.

(iii) If $\mu_n \to \mu$ and $\sigma_n \to \sigma$ along some subsequence, explain why we must then have $P = N(\mu, \sigma^2)$.

(iv) Explain why μ_n and σ_n must converge to finite limits and why P must be some normal distribution (possibly degenerate).

(v) Now suppose (Y_n, Z_n) has a bivariate $N(\mu_n, V_n)$ distribution and $(Y_n, Z_n) \rightsquigarrow Q$ for some probability measure Q on$\mathcal{B}(\mathbb{R}^2)$. Use the univariate reult to explain Q must be a bivraiate normal distribution (possibly degenerate). Hint: What does Cauchy-Schwarz tell you about the off-diagonal terms of V_n?

[4] (truncated CLT) UGMTP Problem 7.21.