Suppose $H$ is a continuously differentiable function on $\mathbb{R}$ which is zero outside some bounded interval. For a given bounded measurable function $f$ on $\mathbb{R}$, define $g(x) := \int f(y)H(x-y)\,dy$.

(i) Show (rigorously) that $g$ is differentiable with $g'(x) = \int f(y)H'(x-y)\,dy$.

(ii) Explain why $g$ belongs to $C^\infty(\mathbb{R})$ if $H \in C^\infty(\mathbb{R})$.

* Suppose $P,P_1,P_2,\ldots$ are probability measures on $\mathcal{B}(\mathbb{R})$ for which $P_n(-\infty,x] \to P(-\infty,x]$ for every $x$ in some dense subset $T$ of $\mathbb{R}$. Show that $P_n \Rightarrow P$ by the following steps.

(i) For each $\epsilon > 0$ show that there exist points $x_1 < x_2 < \cdots < x_k$ in $T$ for which $\max_{i=2}^k |x_i - x_{i-1}| < \epsilon$ and $P(-\infty,x_1] < \epsilon$ and $P(x_k,\infty) < \epsilon$.

(ii) Given $f \in BL(\mathbb{R})$, define a step function $f_k(x) := \sum_{i=2}^k f(x_i)\{x_{i-1} < x \leq x_i\}$. Show that $P_nf_k \to Pf_k$ for each $k$ and that both $P|f-f_k|$ and $\limsup_n P_n|f-f_k|$ are suitably small.

* If you didn’t attempt UGMTP Problem 7.21 last week, try to solve it this week.

* (Root Poisson) UGMTP Problem 7.19. You may assume $\lambda = n \in \mathbb{N}$ if you worry about continuous limits. There is no need to reprove that the Poisson($\lambda$) has both mean and variance equal to $\lambda$.

* Let $X_1, X_2, \ldots$ be iid random variables with $\mathbb{P}\{X_k = +1\} = 1/2 = \mathbb{P}\{X_k = -1\}$. Without any appeals to a CLT, show that $\lim_{n \to \infty} \psi_{Y_n}(t) = \exp(-t^2/2)$ for each real $t$, where $Y_n = n^{-1/2} \sum_{k \leq n} X_k$. 