Statistics 330b/600b, spring 2010
Homework # 2
Due: Thursday 28 January

Please attempt at least the starred problems.

*[1] Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, which is equipped with a σ-field $\mathcal{B}$. Recall that $\sigma(T) := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which T is $\sigma(T)\backslash\mathcal{B}$-measurable. Show that to each f in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a g in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ such that $f = g \circ T$ (that is, $f(x) = g(T(x))$, for all x in $\mathcal{X}$) by following these steps.

(i) If f is the indicator function of $T^{-1}(B)$ and g is the indictor function of B, show that $f = g \circ T$. Hint: If you write f as $\{x \in T^{-1}B\}$ and g as $\{y \in B\}$ there is hardly anything to prove.

(ii) Consider the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(iii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to f. Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property.

(iv) In part (ii), why can't we assume that $\lim g_n(y)$ exists for each y?

*[2] Suppose $f_1, \dots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \dots, \theta_k$ are strictly positive numbers that sum to one. Show that

$$\mu \prod\nolimits_{i \le k} f_i^{\theta_i} \le \prod\nolimits_{i \le k} (\mu f_i)^{\theta_i}$$

by following these steps.

(i) Explain why the inequality is trivially true if $\mu f_i = 0$ for at least one i or if $\mu f_i = \infty$ for at least one i (and all the other μf_j are strictly positive).

(ii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each i and $f_i(x) < \infty$ for each x and i.

(iii) For all $a_1, \dots, a_k \in \mathbb{R}^+$, show that $\prod_{i \le k} a_i^{\theta_i} \le \sum_{i \le k} \theta_i a_i$. Hint: First dispose of the trivial case where at least one a_i is zero, then rewrite the inequality using $b_i = \log a_i$.

(iv) Complete the proof by considering the inequality from (iii) with $a_i = f_i(x)$.

*[3] Suppose a set $\mathcal{E}$ of subsets of $\mathcal{X}$ cannot separate a particular pair of points x, y, that is, for every E in $\mathcal{E}$, either $\{x, y\} \subseteq E$ or $\{x, y\} \subseteq E^c$. Show that $\sigma(\mathcal{E})$ also cannot separate the pair.

[4] The set $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ is called the ***extended real line***. Write $\mathcal{A}$ for the sigma-field on $\overline{\mathbb{R}}$ generated by $\mathcal{B}(\mathbb{R})$ together with the two singleton sets $\{-\infty\}$ and $\{\infty\}$. Show that $\mathcal{A}$ is also generated by $\mathcal{E} = \{[-\infty, t] : t \in \mathbb{R}\}$. *Note: In this problem it is dangerous to write A^c for a subset A of $\mathbb{R}$ because it might not be clear whether you mean $\mathbb{R}\backslash A$ or $\overline{\mathbb{R}}\backslash A$.*