Statistics 330b/600b, spring 2010
Homework # 3
Due: Thursday 4 February

Please attempt at least the starred problems.

*[1] (completeness of $\mathcal{L}^1$) UGMTP Problem 2.18. Note: Don't confuse Cauchy sequences (in $\mathcal{L}^1$ distance) of functions with Cauchy sequences of real numbers.

*[2] A sequence of (real-valued) random variables $\{X_n\}$ is said to converge in probability to a (real-valued) random variable X if $\mathbb{P}\{|X_n - X| > \epsilon\} \to 0$ as $n \to \infty$, for each fixed $\epsilon > 0$.

(i) Suppose $X_n \to 0$ almost surely. For a fixed $\epsilon > 0$ define $Y_n = \{|X_n| > \epsilon\}$. Explain why $Y_n \to 0$ almost surely. Deduce via Dominated Convergence that $\mathbb{P}Y_n \to 0$. Conclude that $X_n \to 0$ in probability.

(ii) Let $\mathbb{P}$ be Lebesgue measure on $\mathcal{B}[0,1)$. If $n = 2^k + j$ with $0 \le j < 2^k$ define X_n to be the indicator function of the interval $[j/2^k, (j+1)/2^k)$. Show that $X_n \to 0$ in probability but X_n does not converge to 0 almost surely.

(iii) Suppose $X_n \to 0$ in probability. Explain why there exists an increasing sequence of positive integers n_k for which $\mathbb{P}\{|X_n| > k^{-1}\} < 2^{-k}$ for all $n \ge n_k$. Deduce that $X_{n_k} \to 0$ almost surely.

*[3] (converse to Borel-Cantelli) UGMTP Problem 2.2. Hint: For part (i) draw a picture of the function $x \mapsto (k - x)(k + 1 - x)$. For part (ii), what happens if $k = k_n$ tends to infinity in such a way that $\mathbb{P}X_n/k_n \to 1$?

[4] (Orlicz norm) UGMTP Problem 2.22. Compare with Problem 2.17 for a special case.

[5] (moments vs. tail decrease) UGMTP Problem 2.8. Hint: Think about the functions $|f|^k$ and $\sum_{k\in\mathbb{N}} n^{k-1}\{|f| \ge n\}$.