Statistics 330b/600b, spring 2010
Homework # 4
Due: Thursday 11 February

Please attempt at least the starred problems.

*[1] Let λ denote the measure on $\mathbb{R}$ with λA equal to the number of points in A (possibly infinite), and let $\mu A = \infty$ for all nonempty A. Show that λ has no density with respect to μ, even though both measures have the same negligible sets.

*[2] Let ν and μ be finite measures on the sigma-field $\sigma(\mathcal{E})$ generated by a field $\mathcal{E}$. Suppose that for each $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\nu E < \epsilon$ for each E in $\mathcal{E}$ with $\mu E < \delta_\epsilon$. Show that ν is absolutely continuous with respect to μ, as measures on $\sigma(\mathcal{E})$, by the following steps.

(i) Suppose $\mu A = 0$ for an A in $\sigma(\mathcal{E})$. For each $\epsilon > 0$, use the result from UGMTP Example 2.5 to show that there exists an increasing sequence of sets $\{E_n\}$ in $\mathcal{E}$ such that $A \subseteq G_\epsilon := \cup_{n\in\mathbb{N}} E_n$ and $\mu G_\epsilon < \delta_\epsilon$.

(ii) Deduce that $\nu A = 0$.

*[3] Suppose X is a real-valued random variable with a Binomial(n, θ) distribution. That is, $\mathbb{P}\{X = k\} = \binom{n}{k}\theta^k(1-\theta)^{n-k}$ for $k = 0, 1, \dots, n$. You may assume these elementary facts: $\mathbb{P}X = n\theta$ and $\mathbb{P}(X - n\theta)^2 = n\theta(1-\theta)$. Let f be a continuous real function defined on $[0, 1]$.

(i) Show that $p_n(\theta) := \mathbb{P}f(X/n)$ is a polynomial in θ.

(ii) Explain why there exists a constant M and, for each $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ for which

$$|f(x) - f(y)| \le \epsilon + M\{|x - y| > \delta_\epsilon\} \qquad \text{for all } x, y \text{ in } [0, 1].$$

(iii) Show that

$$|p_n(\theta) - f(\theta)| \le \mathbb{P}|f(X/n) - f(\theta)| \le \epsilon + \frac{M\theta(1-\theta)}{n\delta_\epsilon^2} < 2\epsilon \qquad \text{if } n > M/(\epsilon\delta_\epsilon^2).$$

(iv) Conclude that for each $\epsilon > 0$ there exists a polynomial $p_\epsilon(\theta)$ for which

$$\sup_{0\le\theta\le1} |f(\theta) - p_\epsilon(\theta)| < \epsilon,$$

a result known as the ***Weierstrass approximation theorem***.

[4] (completeness of $\mathcal{L}^p$ or $\mathcal{L}^\Psi$)) UGMTP Problem 2.19 or 2.23, not both.