Statistics 330b/600b, spring 2010
Homework # 5
Due: Thursday 18 February

Please attempt at least the starred problems.

*[1] Suppose $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces (or metric spaces, if you prefer) with collections of open sets $\mathcal{G}_\mathcal{X}$ and $\mathcal{G}_\mathcal{Y}$. Define $\mathcal{G}_{\mathcal{X}\times\mathcal{Y}}$ to consist of all those subsets of $\mathcal{X}\times\mathcal{Y}$ expressible as unions (not necessarily countable) of sets from $\mathcal{G}_\mathcal{X}\times\mathcal{G}_\mathcal{Y}$. Show that $\mathcal{B}(\mathcal{X}\times\mathcal{Y}) := \sigma(\mathcal{G}_{\mathcal{X}\times\mathcal{Y}}) \supseteq \mathcal{B}(\mathcal{X})\otimes\mathcal{B}(\mathcal{Y})$.

*[2] In class I discussed the Bayesian model where λ is a (prior) probability on $(\Theta, \mathcal{A})$ and $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ is a set of probability measures defined on $(\mathcal{X}, \mathcal{B})$. I assumed each P_θ is absolutely continuous with respect to some (sigma-finite) measure μ on $(\mathcal{X}, \mathcal{A})$, with density $dP_\theta/d\mu = p(x,\theta)$ that is $\mathcal{B}\otimes\mathcal{A}$-measurable.

(i) Define $\mathbb{P} = \lambda\otimes\mathcal{P}$, a probability measure on $\mathcal{B}\otimes\mathcal{A}$. Show that $\mathbb{P}$ has density $p(x,\theta)$ with respect to $\lambda\otimes\mu$.

(ii) Define Q as the $\mathcal{X}$-marginal of $\mathbb{P}$. That is, Q is the image of $\mathbb{P}$ under the map T for which $T(x,\theta) = x$. Show that $dQ/d\mu = q(x) := \lambda^\theta p(x,\theta)$.

(iii) Show that $\mathbb{P}\{(x,\theta) : q(x) = 0\} = Q\{x : q(x) = 0\} = 0$.

(iv) Define

$$p(\theta \mid x) = \begin{cases} p(x,\theta)/q(x) & \text{if } q(x) > 0 \\ 1 & \text{if } q(x) = 0 \end{cases}$$

Define (posterior) probability measures $\Lambda = \{\lambda_x : x \in \mathcal{X}\}$ by $d\lambda_x/d\lambda = p(\theta \mid x)$. Show that $\mathbb{P} = Q\otimes\Lambda$.

*[3] Let $(\mathcal{X}, \mathcal{A}, \mu)$ and $(\mathcal{Y}, \mathcal{B}, \nu)$ be two measure spaces, with both μ and ν sigma-finite. Write $\mathcal{G}$ for the set of all functions expressible as finite linear combinations of measurable rectangles. That is, a typical g in $\mathcal{G}$ is expressible as a finite sum $\sum_{i=1}^k \alpha_i\{x \in A_i,\ y \in B_i\}$ for some sets $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ and real numbers α_i, for $i = 1, 2, \ldots, k$.

Show that for each f in $\mathcal{L}^1(\mathcal{X}\times\mathcal{Y}, \mathcal{A}\otimes\mathcal{B}, \mu\otimes\nu)$ and each $\epsilon > 0$ there exist a $g \in \mathcal{G}$ such that $\mu\otimes\nu|f - g| < \epsilon$. Follow these steps.

(i) First suppose that both μ and ν are finite measures and $|f|$ is bounded. Use a lambda-space argument to establish the asserted approximation property.

(ii) Extend to the sigma-finite case. Hint: First approximate the function f by some $f_n := (-n) \vee (f \wedge n)$.

[4] (quantile minimizes Orlicz distance) UGMTP Problem 4.17.