Throughout this sheet \( \{X_n : n \in \mathbb{N}\} \) will be a fixed exchangeable sequence of random variables, each \( X_i \) taking values in \( \{0,1\} \). That is, for each \( n \), the probability \( \mathbb{P}\{X_1 = x_1, \ldots, X_n = x_n\} \) for \( x_i \in \{0,1\} \) only depends on \( \sum_{i \leq n} x_i \). Equivalently, for each permutation \( \pi \) of \( 1,2,\ldots,n \), the random vectors 
\[
(X_1, X_2, \ldots, X_n) \quad \text{and} \quad ((X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}))
\]

have the same joint distribution.

Define \( S_n := \sum_{i \leq n} X_i \) and \( G_n := \sigma\{S_n, S_{n+1}, \ldots\} \) and \( G_\infty := \cap_{n \in \mathbb{N}} G_n \).

*[1] Show that \( \ldots, (S_n/n, G_n), \ldots, (S_1, G_1) \) is a martingale, by these steps.

(i) For each bounded, \( \mathcal{B}(\mathbb{R}^k) \)-measurable function \( f \), show that
\[
\mathbb{P} X_j f(S_n, S_{n+1}, \ldots, S_{n+k}) = \mathbb{P} X_1 f(S_n, S_{n+1}, \ldots, S_{n+k}) \quad \text{for } j = 1,2,\ldots,n.
\]
Hint: Think of the integrand on the right-hand side as \( g(X_1, X_2, \ldots, X_{n+k}). \) What happens if you interchange \( X_1 \) and \( X_j \)?

(ii) With \( f \) as in part (i), show that
\[
\mathbb{P} S_n f(S_n, S_{n+1}, \ldots, S_{n+k}) = n \mathbb{P} X_1 f(S_n, S_{n+1}, \ldots, S_{n+k})
\]
\[
\mathbb{P} S_{n-1} f(S_n, S_{n+1}, \ldots, S_{n+k}) = (n-1) \mathbb{P} X_1 f(S_n, S_{n+1}, \ldots, S_{n+k})
\]

(iii) Use some sort of generating class argument to establish the asserted martingale property.

*[2] Explain why there exists a \( G_\infty \)-measurable random variable \( Z \), taking values in \( [0,1] \), for which \( S_n/n \to Z \) almost surely.

*[3] For each subset \( J \) of \( \{1,2,\ldots,n\} \) write \( X_J \) for \( \prod_{i \in J} X_i \) and \( |J| \) for the size of \( J \).

(i) For a fixed positive integer \( m \), show that there exist constants \( C_j \) for \( j = 1,2,\ldots,m \), with \( C_m = m! \), such that
\[
S_n^m = \sum_{j=1}^m C_j \sum_{|J|=j} X_J \quad \text{for each } n \geq m.
\]

(ii) Deduce that
\[
\sum_{j=1}^m C_j \binom{n}{j} \mathbb{P}(X_1 \ldots X_j \mid S_n = k) = k^m
\]

(iii) Deduce that \( \mathbb{P}(X_1 \ldots X_m \mid S_n = k) = (k/n)^m + O(1/n) \), with the error bound holding uniformly over \( k = 0,1,\ldots,n \).
Write \( Q \) for the distribution of the random variable \( Z \) from Problem [2].

(i) For each bounded, continuous function \( f \) on \([0,1]\) explain why
\[
P_X X_1 \ldots X_m f(Z) = \lim_{n \to \infty} P X_1 \ldots X_m f(S_n/n) = Q^m f(\theta)
\]

(ii) You may assume that the conditional probability distribution \( P_\theta(\cdot) = P(\cdot | Z = \theta) \) exists. (It does.) Show that \( P_\theta(X_1 \ldots X_m) = \theta^m \) a.e. \([Q]\).

(iii) Deduce that
\[
P_\theta X_1 X_2 \ldots X_m (1 - X_{m+1}) \ldots (1 - X_{m+k}) = \theta^m (1 - \theta)^k
\]

(iv) Explain why \( Q \) is uniquely determined by \( Q^m \) for \( m = 1, 2, \ldots \). Hint: Weierstrass approximation theorem.

For the Polya urn model with the urn initially containing \( r \) red balls and \( b \) black balls, and with \( d = 1 \), show that
\[
P(X_1 \ldots X_m) = \prod_{i=0}^{m-1} \frac{r + i}{r + b + i} = Q^m,
\]
where \( Q \) is the Beta\((\alpha, \beta)\) distribution for some choice of \( \alpha \) and \( \beta \). What does this tell you about the limiting distribution of \( S_n/n \)?