For this sheet, suppose \((\Omega, \mathcal{F}, \mathbb{P})\) and \((\mathcal{I}, \mathcal{B}, \mathbb{Q})\) are two probability spaces and \(T\) is an \(\mathcal{F}\setminus\mathcal{B}\)-measurable map from \(\Omega\) into \(\mathcal{B}\) that has distribution \(\mathbb{Q}\).

*\[1\] Suppose \(\{\mathbb{P}_t : t \in \mathcal{I}\}\) is a set of probability measures on \(\mathcal{I}\) for which \(t \mapsto \mathbb{P}_t F\) is \(\mathcal{B}\)-measurable for each fixed \(F\) in \(\mathcal{I}\).

(i) Show that \(t \mapsto \mathbb{P}_t h(\omega)\) is \(\mathcal{B}\)-measurable for each fixed \(h\) in \(\mathcal{M}^+(\Omega, \mathcal{I})\).

(ii) Suppose \(\mathbb{P}g(T\omega)h(\omega) = Q^t g(t)\mathbb{P}_t^\omega h(\omega)\) for each \(g \in \mathcal{M}^+(\mathcal{I}, \mathcal{B})\) and \(h \in \mathcal{M}^+(\Omega, \mathcal{I})\). Use a lambda-space argument to show that \(\mathbb{P}f(T\omega, \omega) = Q^t \mathbb{P}_t^\omega f(t, \omega)\) for each bounded, \(\mathcal{B} \otimes \mathcal{F}\)-measurable real function \(f\) on \(\mathcal{T} \times \Omega\).

(iii) Suppose the function \(f(t, \omega) := \{(t, \omega) : T\omega \neq t\}\) is product measurable. Explain why \(f(T\omega, \omega) = 0\) for all \(\omega\). Deduce that \(\mathbb{P}_t \{\omega : T\omega \neq t\} = 0\) for \(\mathbb{Q}\) almost all \(t\).

*\[2\] Suppose \(\{\mathbb{P}_t : t \in \mathcal{I}\}\) and \(\{\mathbb{P}_t : t \in \mathcal{I}\}\) are two candidates for the conditional probability distribution of \(\mathbb{P}(\cdot \mid T = t)\). That is,

\[
\mathbb{P}f(\omega) = Q^t \mathbb{P}_t^\omega f(\omega) = Q^t \mathbb{P}_t^\omega f(\omega) \quad \text{for each } f \in \mathcal{M}^+(\Omega, \mathcal{I}).
\]

and \(\mathbb{P}_t \{T \neq t\} = 0 = \mathbb{P}_t \{T \neq t\}\) for \(\mathbb{Q}\) almost all \(t\).

Suppose the sigma-field \(\mathcal{I}\) is countably generated, that is, \(\mathcal{I} = \sigma(\mathcal{E})\) for some countable collection \(\mathcal{E}\) of sets.

(i) Why is there no loss of generality in assuming that \(\mathcal{E}\) is stable under the formation of finite intersections?

(ii) For each \(E \in \mathcal{E}\), show that \(\mathbb{P}_t E = \mathbb{P}_t E\) a.e \([\mathbb{Q}]\). Hint: Consider \(\mathbb{P}\{\omega \in E, T\omega \in B\}\) for various \(B \in \mathcal{B}\).

(iii) Show that there exists a \(\mathbb{Q}\)-negligible set \(N\) such that \(\mathbb{P}_t = \mathbb{P}_t\), as measures on \(\mathcal{I}\), for all \(t \in N^c\).

*\[3\] Suppose \(X_1, \ldots, X_n\) is an exchangeable set of random variables, that is, the joint distribution of \((X_{\pi(1)}, \ldots, X_{\pi(n)})\) is the same for each permutation \(\pi\) of \(\{1, 2, \ldots, n\}\). Define \(S_n = X_1 + \cdots + X_n\) and \(\mathcal{G} = \sigma(S_n)\). Suppose each \(X_i\) is \(\mathbb{P}\)-integrable.

(i) For each bounded \(\mathcal{B}(\mathbb{R})\)-measurable function \(g\), show that \(\mathbb{P}X_1 g(S_n) = \mathbb{P}X_2 g(S_n)\).

(ii) Show that \(\mathbb{P}(X_1 \mid \mathcal{G}) = S_n/n\) almost surely.