Statistics 330b/600b, spring 2010
Homework # 9
Due: Thursday 8 April

For this sheet, suppose $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $(\mathcal{T}, \mathcal{B}, Q)$ *are two probability spaces and* T *is an* $\mathcal{F}\backslash\mathcal{B}$*-measurable map from* Ω *into* $\mathcal{B}$ *that has distribution* Q*.*

*[1] Suppose $\{\mathbb{P}_t : t \in \mathcal{T}\}$ is a set of probability measures on $\mathcal{F}$ for which $t \mapsto \mathbb{P}_t F$ is $\mathcal{B}$-measurable for each fixed F in $\mathcal{F}$.

(i) Show that $t \mapsto \mathbb{P}_t h(\omega)$ is $\mathcal{B}$-measurable for each fixed h in $\mathcal{M}^+(\Omega, \mathcal{F})$.

(ii) Suppose $\mathbb{P} g(T\omega)h(\omega) = Q^t g(t)\mathbb{P}_t^\omega h(\omega)$ for each $g \in \mathcal{M}^+(\mathcal{T}, \mathcal{B})$ and $h \in \mathcal{M}^+(\Omega, \mathcal{F})$. Use a lambda-space argument to show that $\mathbb{P} f(T\omega, \omega) = Q^t \mathbb{P}_t^\omega f(t, \omega)$ for each bounded, $\mathcal{B} \otimes \mathcal{F}$-measurable real function f on $\mathcal{T} \times \Omega$.

(iii) Suppose the function $f(t, \omega) := \{(t, \omega) : T\omega \neq t\}$ is product measurable. Explain why $f(T\omega, \omega) = 0$ for all ω. Deduce that $\mathbb{P}_t\{\omega : T\omega \neq t\} = 0$ for Q almost all t.

*[2] Suppose $\{\mathbb{P}_t : t \in \mathcal{T}\}$ and $\{\widetilde{\mathbb{P}}_t : t \in \mathcal{T}\}$ are two candidates for the conditional probability distribution of $\mathbb{P}(\cdot \mid T = t)$. That is,

$$\mathbb{P} f(\omega) = Q^t \mathbb{P}_t^\omega f(\omega) = Q^t \widetilde{\mathbb{P}}_t^\omega f(\omega) \qquad \text{for each } f \in \mathcal{M}^+(\Omega, \mathcal{F}).$$

and $\mathbb{P}_t\{T \neq t\} = 0 = \widetilde{\mathbb{P}}_t\{T \neq t\}$ for Q almost all t.

Suppose the sigma-field $\mathcal{F}$ is countably generated, that is, $\mathcal{F} = \sigma(\mathcal{E})$ for some countable collection $\mathcal{E}$ of sets.

(i) Why is there no loss of generality in assuming that $\mathcal{E}$ is stable under the formation of finite intersections?

(ii) For each $E \in \mathcal{E}$, show that $\mathbb{P}_t E = \widetilde{\mathbb{P}}_t E$ a.e $[Q]$. Hint: Consider $\mathbb{P}\{\omega \in E, T\omega \in B\}$ for various $B \in \mathcal{B}$.

(iii) Show that there exists a Q-negligible set $\mathcal{N}$ such that $\mathbb{P}_t = \widetilde{\mathbb{P}}_t$, as measures on $\mathcal{F}$, for all $t \in \mathcal{N}^c$.

*[3] Suppose $X_1, \ldots, X_n$ is an exchangeable set of random variables, that is, the joint distribution of $(X_{\pi(1)}, \ldots, X_{\pi(n)})$ is the same for each permutation π of $\{1, 2, \ldots, n\}$. Define $S_n = X_1 + \cdots + X_n$ and $\mathcal{G} = \sigma(S_n)$. Suppose each X_i is $\mathbb{P}$-integrable.

(i) For each bounded $\mathcal{B}(\mathbb{R})$-measurable function g, show that $\mathbb{P} X_1 g(S_n) = \mathbb{P} X_2 g(S_n)$.

(ii) Show that $\mathbb{P}(X_1 \mid \mathcal{G}) = S_n/n$ almost surely.

[4] (Conditional Jensen) UGMTP Problem 5.13.