Chapter 2

A modicum of measure theory

2 February 2004: Modification of Section 2.11.

*1. Generating classes of functions

Theorem <Dynkin.thm> is often used as the starting point for proving facts about measurable functions. One first invokes the Theorem to establish a property for sets in a sigma-field, then one extends by taking limits of simple functions to $\mathcal{N}^+$ and beyond, using Monotone Convergence and linearity arguments. Sometimes it is simpler to invoke an analog of the $\lambda$-system property for classes of functions.

<1> Definition. Let $\mathcal{H}$ be a set of bounded, real-valued functions on a set $X$. Call $\mathcal{H}$ a $\lambda$-space if:

(i) $\mathcal{H}$ is a vector space
(ii) each constant function belongs to $\mathcal{H}$;
(iii) if $\{h_n\}$ is an increasing sequence of functions in $\mathcal{H}$ whose pointwise limit $h$ is bounded then $h \in \mathcal{H}$.

The sigma-field properties of $\lambda$-spaces are slightly harder to establish than their $\lambda$-system analogs, but the reward of more streamlined proofs will make the extra, one-time effort worthwhile. First we need an analog of the fact that a $\lambda$-system that is stable under finite intersections is also a sigma-field.

Remember that $\sigma(\mathcal{H})$ is the smallest sigma-field on $X$ for which each $h$ in $\mathcal{H}$ is $\sigma(\mathcal{H})\cap \mathcal{B}(\mathbb{R})$-measurable. It is the sigma-field generated by the collection of sets $\{h \in B\}$ with $h \in \mathcal{H}$ and $B \in \mathcal{B}(\mathbb{R})$. It is also generated by

$$E_{\mathcal{H}} := \{ \{h < c\} : h \in \mathcal{H}, c \in \mathbb{R}\}.$$

<2> Lemma. If a $\lambda$-space $\mathcal{H}$ is stable under the formation of pointwise products of pairs of functions then it consists of all bounded, $\sigma(\mathcal{H})$-measurable functions.

Proof. By definition, every function in $\mathcal{H}$ is $\sigma(\mathcal{H})$-measurable. The proof that every bounded, $\sigma(\mathcal{H})$-measurable function belongs to $\mathcal{H}$ will follow from the following four facts:

(a) $\mathcal{H}$ is stable under uniform limits
(b) if $h_1$ and $h_2$ are in $\mathcal{H}$ then so are $h_1 \vee h_2$ and $h_1 \wedge h_2$
(c) the collection of sets \( A_0 := \{A \in \mathcal{A} : A \in \mathcal{H}\} \) is a \( \sigma \)-field

(d) \( \mathcal{E}_\mathcal{H} \subseteq A_0 \) and hence \( \sigma(\mathcal{H}) = \sigma(\mathcal{E}_\mathcal{H}) \subseteq A_0 \)

For suppose \( g \) is a bounded, \( \sigma(\mathcal{H}) \)-measurable function. With no loss of generality (or by means of some linear rescaling) we may assume that \( 0 \leq g \leq 1 \). For each real \( c \), the (indicator function of the) \( \sigma(\mathcal{H}) \)-measurable set \( \{ g \geq c \} \) belongs to \( \mathcal{H} \), by virtue of (d) and (c). The vector space property of \( \mathcal{H} \) ensures that the simple function \( g_n := 2^{-n} \sum_{i=1}^{2^n} (g \geq i/2^n) \) also belongs to \( \mathcal{H} \). Stability of \( \mathcal{H} \) under uniform limits then implies that \( g \in \mathcal{H} \).

**Proof of (a).** Suppose \( h_n \to h \) uniformly, with \( h_n \in \mathcal{H} \). Write \( \delta_n \) for \( 2^{-n} \). With no loss of generality we may suppose \( h_n + \delta_n \geq h \geq h_n - \delta_n \) for all \( n \). Notice that

\[
    h_n + 3\delta_n = h_n + \delta_n + \delta_{n-1} \geq h + \delta_{n-1} \geq h_{n-1}.
\]

the functions \( g_n := h_n + 3(\delta_1 + \ldots + \delta_n) \) all belong to \( \mathcal{H} \), and \( g_n \uparrow h + 3 \). It follows that \( h + 3 \in \mathcal{H} \), and hence, \( h \in \mathcal{H} \).

**Proof of (b).** It is enough if we show that \( h^+ \in \mathcal{H} \) for each \( h \in \mathcal{H} \), because \( h_1 \lor h_2 = h_1 + (h_2 - h_1)[^+ \lor \, - (h_1 \land h_2) = (-h_1) \lor (-h_2) \). Suppose \( c \leq h \leq d \), for constants \( c \) and \( d \). First note that, for every polynomial \( p(y) = a_0 + a_1y + \ldots + a_my^m \), we have

\[
    p(h) = a_0 + a_1h + \ldots + a_my^m \in \mathcal{H},
\]

because the constant function \( a_0 \) and each of the powers \( h^k \) belong to the vector space \( \mathcal{H} \). By a minor extension of the Weierstrass approximation result from Problem [WEIERSTRASS], the continuous function \( y \mapsto y^+ \) can be uniformly approximated by a polynomial on the interval \( [c, d] \). That is, there exists a sequence of polynomials \( p_n \) such that \( \sup_{c \leq y \leq d} |p_n(y) - y^+| \to 0 \) as \( n \to \infty \). In particular, \( h^+ \) is a uniform limit of \( p_n(h) \), so that \( h^+ \in \mathcal{H} \) by virtue of (a).

**Proof of (c).** The fact that \( 1 \in \mathcal{H} \) and the stability of \( \mathcal{H} \) under monotone limits, differences, and finite products implies that \( \mathcal{A}_0 \) is a \( \lambda \)-system of sets that is stable under finite intersections, that is, \( \mathcal{A}_0 \) is a \( \sigma \)-field.

**Proof of (d).** Suppose \( h \in \mathcal{H} \) and \( c \in \mathbb{R} \). By (b), the function

\[
    h_0 := (1 + h - c)^+ \land 1
\]

belongs to \( \mathcal{H} \). Notice that \( 0 \leq h_0 \leq 1 \) and \( \{h_0 = 1\} = \{h \geq c\} \). As a monotone increasing limit of functions \( 1 - h_0^n \) from \( \mathcal{H} \), the (indicator function of the) set \( \{ h < c \} \) also belongs to \( \mathcal{H} \). \( \square \)

**Theorem.** Let \( \mathcal{G} \) be a set of functions from a \( \lambda \)-space \( \mathcal{H} \). If \( \mathcal{G} \) is stable under the formation of pointwise products of pairs of functions then \( \mathcal{H} \) contains all bounded, \( \sigma(\mathcal{G}) \)-measurable functions.

**Proof.** Let \( \mathcal{H}_0 \) be the smallest \( \lambda \)-space containing \( \mathcal{G} \). By Lemma <2>, it is enough to show that \( \mathcal{H}_0 \) is stable under pairwise products.
2.1 Generating classes of functions

Argue as in Theorem <Dynkin.thm> for \(\lambda\)-systems of sets. An almost routine calculation shows that \(\mathcal{H}_1 := \{ h \in \mathcal{H}_0 : hg \in \mathcal{H}_0 \text{ for all } g \in \mathcal{G} \} \) is a \(\lambda\)-space containing \(\mathcal{G}\). The only subtlety lies in showing that \(\mathcal{H}_1\) is stable under monotone increasing limits. If \(h_n \in \mathcal{H}_1\) and \(h_n \uparrow h\) and \(g \geq 0\), then \(gh_n \uparrow gh\). At points where \(g\) is strictly negative, the sequence \(gh_n\) would not be increasing. However, we can find a constant \(C\) large enough that \(g + C \geq 0\) everywhere, and hence \(gh\) belongs to \(\mathcal{H}_0\) as a monotone increasing limit of \(\mathcal{H}_0\) functions \(h_n g + Ch_n - Ch\). It follows that \(\mathcal{H}_1 = \mathcal{H}_0\). That is, \(h_0 g \in \mathcal{H}_0\) for all \(h_0 \in \mathcal{H}_0\) and \(g \in \mathcal{G}\).

Similarly, \(\mathcal{H}_2 := \{ h \in \mathcal{H}_0 : hoh \in \mathcal{H}_0 \text{ for all } h_0 \text{ in } \mathcal{H}_0 \} \) is a \(\lambda\)-space. By the result for \(\mathcal{H}_1\) we have \(\mathcal{H}_2 \supseteq \mathcal{G}\), and hence \(\mathcal{H}_2 = \mathcal{H}_0\). That is, \(\mathcal{H}_0\) is stable under products.

Exercise. Let \(\mu\) be a finite measure on \(\mathcal{B}(\mathbb{R}^k)\). Write \(\mathcal{C}_0\) for the vector space of all continuous real functions on \(\mathbb{R}^k\) with compact support. Suppose \(f\) belongs to \(L^1(\mu)\). Show that for each \(\epsilon > 0\) there exists a \(g\) in \(\mathcal{C}_0\) such that \(\mu|f - g| < \epsilon\).

Solution: Define \(\mathcal{H}\) as the collection of all bounded functions in \(L^1(\mu)\) that can be approximated arbitrarily closely (in \(L^1(\mu)\) norm) by functions from \(\mathcal{C}_0\). Check that \(\mathcal{H}\) is a \(\lambda\)-space. Trivially it contains \(\mathcal{C}_0\). The sigma-field \(\sigma(\mathcal{C}_0)\) coincides with the Borel sigma-field. Why? The class \(\mathcal{H}\) consists of all bounded, nonnegative Borel measurable functions.

See Problem [C0.Dense2] for the extension of the approximation result to infinite measures.