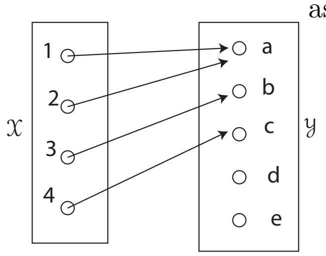
**Statistics 330b/600b, Math 330b spring 2011**
Homework # 1
Due: Thursday 20 January

*Please attempt at least the starred problems.*

*[1] Suppose $T$ maps a set $\mathcal{X}$ into a set $\mathcal{Y}$. For each $B \subseteq \mathcal{Y}$ and $A \subseteq \mathcal{X}$ define $T^{-1}B := \{x \in \mathcal{X} : T(x) \in B\}$. and $T(A) := \{T(x) : x \in A\}$. Some of the following eight assertions are true in general and some are false.




$$
\begin{array}{ll}
(i) \quad T\big(\cup_i A_i\big) = \cup_i T(A_i) & (ii) \quad T^{-1}\big(\cup_i B_i\big) = \cup_i T^{-1}(B_i) \\
(iii) \quad T\big(\cap_i A_i\big) = \cap_i T(A_i) & (iv) \quad T^{-1}\big(\cap_i B_i\big) = \cap_i T^{-1}(B_i) \\
(v) \quad T\left(A^c\right) = \left(T\left(A\right)\right)^c & (vi) \quad T^{-1}\left(B^c\right) = \left(T^{-1}\left(B\right)\right)^c \\
(vii) \quad T^{-1}\big(T(A)\big) = A & (viii) \quad T\big(T^{-1}(B)\big) = B
\end{array}
$$

Provide counterexamples for each of the false assertions. You might find it helpful to contemplate the picture for a special case.

*[2] Suppose a set $\mathcal{E}$ of subsets of $\mathcal{X}$ cannot separate a particular pair of points $x$, $y$, that is, for every $E$ in $\mathcal{E}$, either $\{x, y\} \subseteq E$ or $\{x, y\} \subseteq E^c$. Show that $\sigma(\mathcal{E})$ also cannot separate the pair. Hint: Consider $\mathcal{D} := \{A \in \sigma(\mathcal{E}) : A \text{ does not separate } x \text{ and } y\ \}$.

[3] Follow these steps to show that there cannot exist a translation invariant (countably additive) probability measure defined for the collection of all subsets of $(0, 1]$. For $a$ and $y$ in $(0, 1]$ define

$$
x \oplus y = \begin{cases} x + y & \text{if } x + y \le 1 \\ x + y - 1 & \text{if } x + y > 1 \end{cases} .
$$

For $A \subseteq (0, 1]$ and $x \in (0, 1]$ define $A \oplus x = \{y \oplus x : y \in A\}$. Suppose $\mu$ is a measure that is defined for all subsets of $(0, 1]$ for which $\mu\big(A \oplus x\big) = \mu A$ for all $A$ and $x$. Define $\mathcal{Q} = \{q \in (0, 1] : q \text{ is rational }\}$. Define an equivalence relation on $(0, 1]$ by $x \sim y$ if $x = y \oplus r$ for some $r \in \mathcal{R}$. Let $A$ be a set that contains exactly one point from each equivalence class.

(i) Show that $A \oplus r$ and $A \oplus s$ are disjoint sets if $r$ and $s$ are distinct elements of $\mathcal{Q}$ and that $(0, 1] = \cup_{r \in \mathcal{Q}} \big(A \oplus r\big)$. Hint: The argument is much neater if you identify $(0, 1]$ with the perimeter of a circle of circumference 1.

(ii) Deduce that $\mu(0, 1] = \sum_{r \in \mathcal{Q}} \mu\big(A \oplus r\big)$.

(iii) Show that

$$
\mu(0, 1] = \begin{cases} \infty & \text{if } \mu A > 0 \\ 0 & \text{if } \mu A = 0 \end{cases} .
$$

Deduce that $\mu$ cannot be a probability measure (or any other non-trivial, finite measure).