Statistics 330b/600b, Math 330b spring 2011
Homework # 10
Due: Thursday 7 April

Please attempt at least the starred problems.

*[1] (conditional Jensen) UGMTP Problem 5.13.

*[2] Suppose $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$. Let $X_\mathcal{G}$ be a version of $\mathbb{P}_\mathcal{G} X$. Define $\text{var}_\mathcal{G}(X)$ to equal $\mathbb{P}_\mathcal{G}(X - X_\mathcal{G})^2$. Show that

$$\text{var}(X) = \mathbb{P}\big(\text{var}_\mathcal{G} X\big) + \text{var}(\mathbb{P}_\mathcal{G} X).$$

Remark: You could just expand both sides, but it is more instructive to argue first that, without loss of generality, $\mathbb{P}X = 0$. You should then recognize a familiar $\mathcal{L}^2$ fact.

*[3] (An alternative to the method described in UGMTP §6.6.) Suppose $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a filtration on Ω (an increasing sequence of sub-sigma-fields of $\mathcal{F}$). Define $\mathcal{F}_\infty := \sigma(\mathcal{E})$ where $\mathcal{E} := \cup_{n \in \mathbb{N}} \mathcal{F}_n$. For n in $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ define $X_n := \mathbb{P}_{\mathcal{F}_n} X$. Prove that $X_n \to X_\infty$ both almost surely and in $\mathcal{L}^1$ norm by the following steps.

(i) Explain why there is no loss of generality in assuming $X \geq 0$.

(ii) Explain why there exists an $\mathcal{F}_\infty$-measurable random variable Z for which $X_n \to Z$ almost surely and $\mathbb{P}Z \leq \mathbb{P}X$.

(iii) Temporarily suppose $X \leq C$ for a finite constant C. Explain why $\mathbb{P}|X_n - Z| \to 0$. For each F in $\mathcal{E}$, explain why $\mathbb{P}XF = \mathbb{P}X_n F$ for all large enough n. Explain why $\mathbb{P}X_n F \to \mathbb{P}ZF$. Deduce (via a generating class argument) that $Z = X_\infty$ almost surely.

(iv) For an unbounded X, explain why $X_n \geq \mathbb{P}_{\mathcal{F}_n}(X \wedge C) \to \mathbb{P}_{\mathcal{F}_\infty}(X \wedge C)$ almost surely, for each finite constant C. Deduce that $Z \geq \mathbb{P}_{\mathcal{F}_\infty}(X \wedge C)$ almost surely.

(v) Deduce that $\mathbb{P}Z = \mathbb{P}X$. Explain why $\mathbb{P}|X_n - Z| = 2\mathbb{P}(Z - X_n)^+ \to 0$.

(vi) Explain why $Z = X_\infty$ almost surely.

[4] (Neyman factorization theorem cf. UGMTP Example 5.31) Suppose $\mathbb{P}$ and $\mathbb{P}_\theta$, for $\theta \in \Theta$, are probability measures defined on a sigma-field $\mathcal{F}$, for some index set Θ. Suppose also that $\mathcal{G}$ is a sub-sigma-field of $\mathcal{F}$ and that there exist versions of densities

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}} = g_\theta(\omega) h(\omega) \qquad \text{with } g_\theta \in \mathcal{M}^+(\mathcal{G}) \text{ for each } \theta$$

for a fixed $h \in \mathcal{M}^+(\mathcal{F})$ that doesn't depend on θ.

(i) Define H to be a version of $\mathbb{P}_\mathcal{G} h$. [That is, choose one H from the $\mathbb{P}$-equivalence class of possibilities.] Show that $\mathbb{P}_\theta\{H = 0\} = 0 = \mathbb{P}_\theta\{H = \infty\}$ for each θ.

(ii) For each X in $\mathcal{M}^+(\mathcal{F})$, show that there exists a version of the conditional expectation $\mathbb{P}_\theta(X \mid \mathcal{G})$ that doesn't depend on θ, namely,

$$\mathbb{P}_\theta(X \mid \mathcal{G}) = \frac{\mathbb{P}_\mathcal{G}(Xh)}{H}\{0 < H < \infty\} \qquad \text{a.e. } [\mathbb{P}_\theta] \text{ for every } \theta.$$