Statistics 330b/600b, Math 330b spring 2011
Homework # 11
Due: Thursday 14 April

Please attempt at least the starred problems.

*[1] Suppose $X_n \to X$ almost surely, with $|X_n| \le H$ for an H in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a sub-sigma-field of $\mathcal{F}$. Show that $\mathbb{P}_{\mathcal{G}} X_n \to \mathbb{P}_{\mathcal{G}} X$ almost surely. Hint: You could imitate the proof of Dominated Convergence given in class. You would first need to establish the analog of "Monotone Convergence down".

[2] Let $(\mathcal{X}, d)$ be a metric space with a countable dense subset $\{x_i : i \in \mathbb{N}\}$. For a fixed $\epsilon > 0$ let B_i denote the open ball of radius ϵ and center x_i. Define functions $g_0 \equiv \epsilon$ and $g_i(x) = (1 - d(x, B_i))^+$. Define $G_N(x) := \sum_{0 \le i \le N} g_i(x)$ and $\ell_{i,N}(x) = g_i(x)/G_N(x)$ for $0 \le i \le N$. Note that $\sum_{0 \le i \le N} \ell_{i,N} \equiv 1$.

(i) Show that $\ell_{i,N} \in \text{BL}(\mathcal{X})$ for all i and N.

(ii) Show that $\ell_{0,N}(x) \downarrow L_\epsilon(x)$ with $L_\epsilon(x) < \epsilon$ for each x.

(iii) Show that $\text{diam}\left(\{\ell_{i,N} > 0\}\right) \le 4\epsilon$ for each $i \ge 1$.

*[3] Suppose $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are both separable metric spaces. Define a metric on $\mathcal{X} \times \mathcal{Y}$ by

$$d\left((x_1, y_1), (x_2, y_2)\right) := \max\left(d_{\mathcal{X}}(x_1, x_2), d_{\mathcal{Y}}(y_1, y_2)\right)$$

Remark. By UGMTP Problem 4.6, $\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y}) = \mathcal{B}(\mathcal{X} \times \mathcal{Y})$. You may assume this fact.

Suppose $\{X_n : n \in \mathbb{N}\}$ are random elements of $\mathcal{X}$ and $\{Y_n : n \in \mathbb{N}\}$ are random elements of $\mathcal{Y}$. Suppose μ is a probability measure on $\mathcal{B}(\mathcal{X} \times \mathcal{Y})$. Suppose also that

$$\mathbb{P} g(X_n) h(Y_n) \to \mu g(x) h(y) \qquad \text{for all } g \in \text{BL}(\mathcal{X}, d_{\mathcal{X}}) \text{ and } h \in \text{BL}(\mathcal{Y}, d_{\mathcal{Y}}).$$

Follow these steps to prove that $(X_n, Y_n) \rightsquigarrow \mu$.

(i) Fix an $\epsilon > 0$. Define $\ell_{i,N}$ as in Problem 2. For each f in $\text{BL}(\mathcal{X} \times \mathcal{Y}, d)$ show that

$$|f(x, y) - \sum\nolimits_{1 \le i \le N} f(x_i, y) \ell_{i,N}(x)| \le \|f\|_{\text{BL}} \left(\ell_{0,N}(x) + 4\epsilon\right)$$

(ii) For an appropriate choice of N, deduce that

$$\limsup_n |\mathbb{P} f(X_n, Y_n) - \mu f| \le C\epsilon$$

for some constant C.

(iii) Complete the proof.

*[4] Suppose $(X_n, \mathcal{F}_n) : n = 0, 1, \ldots, n\}$ is a martingale with $X_0 = 0$ and increments $\xi_i = X_i - X_{i-1}$ for which there exist finite constants a_i, b_i with $a_i \le \xi_i \le b_i$ almost surely. Prove the Hoeffding inequality for martingales: for each $y > 0$,

$$\mathbb{P}\{X_n \ge y\} \le \exp\left(-2y^2/K\right) \qquad \text{where } K = \sum\nolimits_{i=1}^n (B_i - a_i)^2$$

Follow these steps.

(i) For each $\theta > 0$ and constants $a < b$, show that

$$e^{x\theta} \le \frac{b-x}{b-a} e^{\theta a} + \frac{x-a}{b-a} e^{\theta b} \qquad \text{for all } a \le x \le b.$$

(ii) For constants $a < 0 < b$, prove that

$$\frac{b}{b-a} e^{a\theta} - \frac{a}{b-a} e^{b\theta} \le \exp\left(\frac{1}{8}\theta^2 (b-a)^2\right) \qquad \text{for all } \theta > 0.$$

Hint: Consider a random variable Y for which $\mathbb{P}\{Y = a\} + \mathbb{P}\{Y = b\} = 1$ and $\mathbb{P}Y = 0$. You may use results proved in class.

(iii) If Z is a random variable with $\mathbb{P}_{\mathcal{G}} Z = 0$ and $a \le Z \le b$ almost surely, show that

$$\mathbb{P}_{\mathcal{G}} e^{\theta Z} \le \exp\left(\frac{1}{8}\theta^2 (b-a)^2\right) \qquad \text{almost surely.}$$

(iv) Complete the proof.