**Statistics 330b/600b, Math 330b spring 2011**
Homework # 2
Due: Thursday 27 January

*Please attempt at least the starred problems.*

*[1] Suppose $T : \mathcal{X} \to \mathcal{Y}$. Let $\mathcal{A}$ be a sigma-field of subsets of $\mathcal{X}$ and $\mathcal{B}$ be a sigma-field of subsets of $\mathcal{Y}$.

(i) Show that $\mathcal{B}_0 := \{B \subseteq \mathcal{Y} : T^{-1}(B) \in \mathcal{A}\}$ is the largest sigma-field on $\mathcal{Y}$ for which $T$ is $\mathcal{A}\backslash\mathcal{B}_0$-measurable.

(ii) Show that $\mathcal{A}_0 := \{T^{-1}B : B \in \mathcal{B}\}$ is the smallest sigma-field on $\mathcal{X}$ for which $T$ is $\mathcal{A}_0\backslash\mathcal{B}$-measurable. *[Note: The sigma-field $\mathcal{A}_0$ is often denoted by $\sigma(T)$ without explit mention of $\mathcal{B}$.]*

*[2] Let $T$ and $\sigma(T)$ be as in Problem [1]. Show that to each $f$ in $\mathcal{M}^+(\mathcal{X}, \sigma(T))$ there exists a $g$ in $\mathcal{M}^+(\mathcal{Y}, \mathcal{B})$ such that $f = g \circ T$ (that is, $f(x) = g(T(x))$, for all $x$ in $\mathcal{X}$) by following these steps.

(i) If $f$ is the indicator function of $T^{-1}(B)$ and $g$ is the indictor function of $B$, show that $f = g \circ T$. Hint: If you write $f$ as $\{x \in T^{-1}B\}$ and $g$ as $\{y \in B\}$ there is hardly anything to prove.

(ii) Consider the case where $f \in \mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$.

(iii) Suppose $f_n = g_n \circ T$ is a sequence in $\mathcal{M}^+_{\text{simple}}(\mathcal{X}, \sigma(T))$ that increases pointwise to $f$. Define $g(y) = \limsup g_n(y)$ for each $y$ in $\mathcal{Y}$. Show that $g$ has the desired property.

(iv) In part (iii), why can't we assume that $\lim g_n(y)$ exists for each $y$ in $\mathcal{Y}$?

*[3] Suppose $f_1, \dots, f_k \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\theta_1, \dots, \theta_k$ are strictly positive numbers that sum to one. Let $\mu$ be a measure on $\mathcal{A}$. Show that

$$\mu \prod\nolimits_{i \le k} f_i^{\theta_i} \le \prod\nolimits_{i \le k} (\mu f_i)^{\theta_i}$$

by following these steps.

(i) Explain why the inequality is trivially true if $\mu f_i = 0$ for at least one $i$ or if $\mu f_i = \infty$ for at least one $i$ (and all the other $\mu f_j$ are strictly positive).

(ii) Explain why there is no loss of generality in assuming that $\mu f_i = 1$ for each $i$ and $f_i(x) < \infty$ for each $x$ and $i$.

(iii) For all $a_1, \dots, a_k \in \mathbb{R}^+$, show that $\prod_{i \le k} a_i^{\theta_i} \le \sum_{i \le k} \theta_i a_i$. Hint: First dispose of the trivial case where at least one $a_i$ is zero, then rewrite the inequality using $b_i = \log a_i$.

(iv) Complete the proof by considering the inequality from (iii) with $a_i = f_i(x)$.

[4] (inner/outer regularity) UGMTP Problem 2.12.