Statistics 330b/600b, Math 330b spring 2011
Homework # 3
Due: Thursday 3 February

Please attempt at least the starred problems.

*[1] For f in $\mathcal{L}^1(\mu)$ define $\|f\|_1 = \mu|f|$. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}^1(\mu)$, that is, $\|f_n - f_m\|_1 \to 0$ as $\min(m,n) \to \infty$. Show that there exists an f in $\mathcal{L}^1(\mu)$ for which $\|f_n - f\|_1 \to 0$, by following these steps. Note: Don't confuse Cauchy sequences (in $\mathcal{L}^1$ distance) of functions with Cauchy sequences of real numbers.

(i) Find an increasing sequence $\{n(k)\}$ such that $\sum_{k=1}^{\infty} \left\|f_{n(k)} - f_{n(k+1)}\right\|_1 < \infty$. Deduce that the function $H := \sum_{k=1}^{\infty} |f_{n(k)} - f_{n(k+1)}|$ is integrable.

(ii) Show that there exists a real-valued, measurable function f for which

$$H \ge |f_{n(k)}(x) - f(x)| \to 0 \qquad \text{as } k \to \infty, \text{ for } \mu \text{ almost all } x.$$

Deduce that $\left\|f_{n(k)} - f\right\|_1 \to 0$ as $k \to \infty$.

(iii) Show that f belongs to $\mathcal{L}^1(\mu)$ and $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

*[2] (Orlicz norms) UGMTP Problem 2.22.

*[3] For each convex, real valued function Ψ on the real line there exists a countable family of linear functions for which $\Psi(x) = \sup_{i \in \mathbb{N}}(a_i + b_i x)$ for all x (see Appendix C of UGMTP). Use this representation to prove Jensen's inequality: if $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}$ a probability measure, then $\mathbb{P}\Psi(X) \ge \Psi(\mathbb{P}X)$. Hint: If $Y \ge Z$ and $\mathbb{P}Z^- < \infty$ then $\mathbb{P}Y^- < \infty$.

[4] Suppose $\mathcal{A}$ is a sigma-field of subsets of $\mathcal{X}$ and $\mathcal{N}$ is a set of subsets of $\mathcal{X}$ that is stable under countable unions. Define

$$\mathcal{B} := \{B \subseteq \mathcal{X} : |B - A| \le N \text{ for some } A \in \mathcal{A} \text{ and some } N \in \mathcal{N}\}.$$

Note: $|B - A| \le N$ could also be written $A \Delta B \subseteq N$, but that form is harder to work with.

(i) Show that $\mathcal{B}$ is a sigma-field.

(ii) If μ is a countably additive measure on $\mathcal{A}$, show that $\mathcal{N}_\mu := \{N \in \mathcal{A} : \mu N = 0\}$ is stable under countable unions.

(iii) With $\mathcal{N} = \mathcal{N}_\mu$ and $\mathcal{B}$ as in (i), show that μ has a unique extension to a countably additive measure $\widetilde{\mu}$ on $\mathcal{B}$ with $\widetilde{\mu}B = \mu A$ if $A \in \mathcal{A}$ and $|B - A| \le N \in \mathcal{N}_\mu$.

[5] For each θ in $[0,1]$ let $P_{n,\theta}$ be the Binomial(n,θ) distribution. That is,

$$P_{n,\theta}\{k\} = \binom{n}{k}\theta^k(1-\theta)^{n-k} \qquad \text{for } k = 0, 1, \ldots, n.$$

You may assume these elementary facts:

$$P^x_{n,\theta} x = \sum_{k=0}^n k P_{n,\theta}\{k\} = n\theta$$
$$P^x_{n,\theta}(x - n\theta)^2 = \sum_{k=0}^n (k - n\theta)^2 P_{n,\theta}\{k\} = n\theta(1-\theta)$$

Let f be a continuous function defined on $[0,1]$. Remember that f is also uniformly continuous: for each fixed ϵ there exists a $\delta > 0$ such that

$$(*) \qquad |f(s) - f(t)| \le \epsilon \qquad \text{whenever } |s-t| \le \delta, \text{ for } s,\, t \text{ in } [0,1].$$

Remember also that $|f|$ must be uniformly bounded.

(i) Show that $p_n(\theta) := P^x_{n,\theta} f(x/n)$ is a polynomial in θ.

(ii) Suppose $|f| \le M$, for a constant M. Show that

$$|f(x/n) - f(\theta)| \le \epsilon + 2M\{\,|(x/n) - \theta| > \delta\} \le \epsilon + \frac{2M|x - n\theta|^2}{n^2\delta^2}.$$

where ϵ and δ are as in $(*)$.

(iii) Deduce that $\sup_{0\le\theta\le1} |p_n(\theta) - f(\theta)| < 2\epsilon$ for n large enough. That is, deduce that $f(\cdot)$ can be uniformly approximated by polynomials over the range $[0,1]$, a result known as the ***Weierstrass approximation theorem***.