Statistics 330b/600b, Math 330b spring 2011
Homework # 4
Due: Thursday 10 February

Please attempt at least the starred problems.

*[1] Suppose $f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$ and $\mu f < \infty$. Show that for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(fA) < \epsilon$ for every $A \in \mathcal{A}$ such that $\mu A < \delta$. Hint: If the assertion were false, there would exist an $\epsilon > 0$ and a sequence of sets with $\mu A_n < 2^{-n}$ but $\mu(fA_n) \geq \epsilon$.

[2] Suppose $f \in \mathcal{L}^1(\mathcal{X}, \mathcal{A}, \mu)$. Show that $|\mu f| \leq \mu|f|$.

*[3] Suppose $f_1, f_2 \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$. Show that

$$\sqrt{(\mu f_1)^2 + (\mu f_2)^2} \leq \mu\sqrt{f_1^2 + f_2^2}$$

Hint: Define $F = \sqrt{f_1^2 + f_2^2}$. Explain why, without loss of generality, we may assume $\mu F = 1$. Define a probability measure P by $dP/d\mu = F$. Reduce the asserted inequality to an analogous inequality involving P and the functions $g_i := f_i\{F > 0\}/F$.

*[4] Let $A_1, A_2, \dots$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $X_n = A_1 + \cdots + A_n$ and $\sigma_n = \mathbb{P}X_n$. Suppose $\sigma_n \to \infty$ and $\|X_n/\sigma_n\|_2 \to 1$. *(Compare with the inequality $\|X_n/\sigma_n\|_2 \geq 1$, which follows from Jensen's inequality.)*

(i) Show that

$$\{X_n = 0\} \leq \frac{(k - X_n)(k + 1 - X_n)}{k(k+1)}$$

for each positive integer k. Hint: Draw a picture.

(ii) By an appropriate choice of k (depending on n) in (i), deduce that $\sum_1^\infty A_i \geq 1$ almost surely.

(iii) Prove that $\sum_m^\infty A_i \geq 1$ almost surely, for each fixed m. Hint: Show that the two convergence assumptions also hold for the sequence $A_m, A_{m+1}, \dots$.

(iv) Deduce that $\mathbb{P}\{\omega \in A_i \text{ i. o. }\} = 1$.

(v) If $\{B_i\}$ is a sequence of events for which $\sum_i \mathbb{P}B_i = \infty$ and $\mathbb{P}B_iB_j = \mathbb{P}B_i\mathbb{P}B_j$ for $i \neq j$, show that $\mathbb{P}\{\omega \in B_i \text{ i. o. }\} = 1$.

[5] Adapt the method from HW 3.1 to prove completeness of $\mathcal{L}^p(\mathcal{X}, \mathcal{A}, \mu)$ for a fixed $p > 1$. Argue as follows.

(i) First show that the triangle inequality holds for infinite sums of functions in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$: $\left\|\sum_{i\in\mathbb{N}} g_i\right\|_p \leq \sum_{i\in\mathbb{N}} \|g_i\|_p$. You may assume the result from HW 3.2.

(ii) Justify replacement of $\mathcal{L}^1$ by $\mathcal{L}^p$ in the argument from HW 3.1. See also UGMTP Problem 2.19.

If this problem is too easy, try UGMTP PRoblem 2.23 instead.